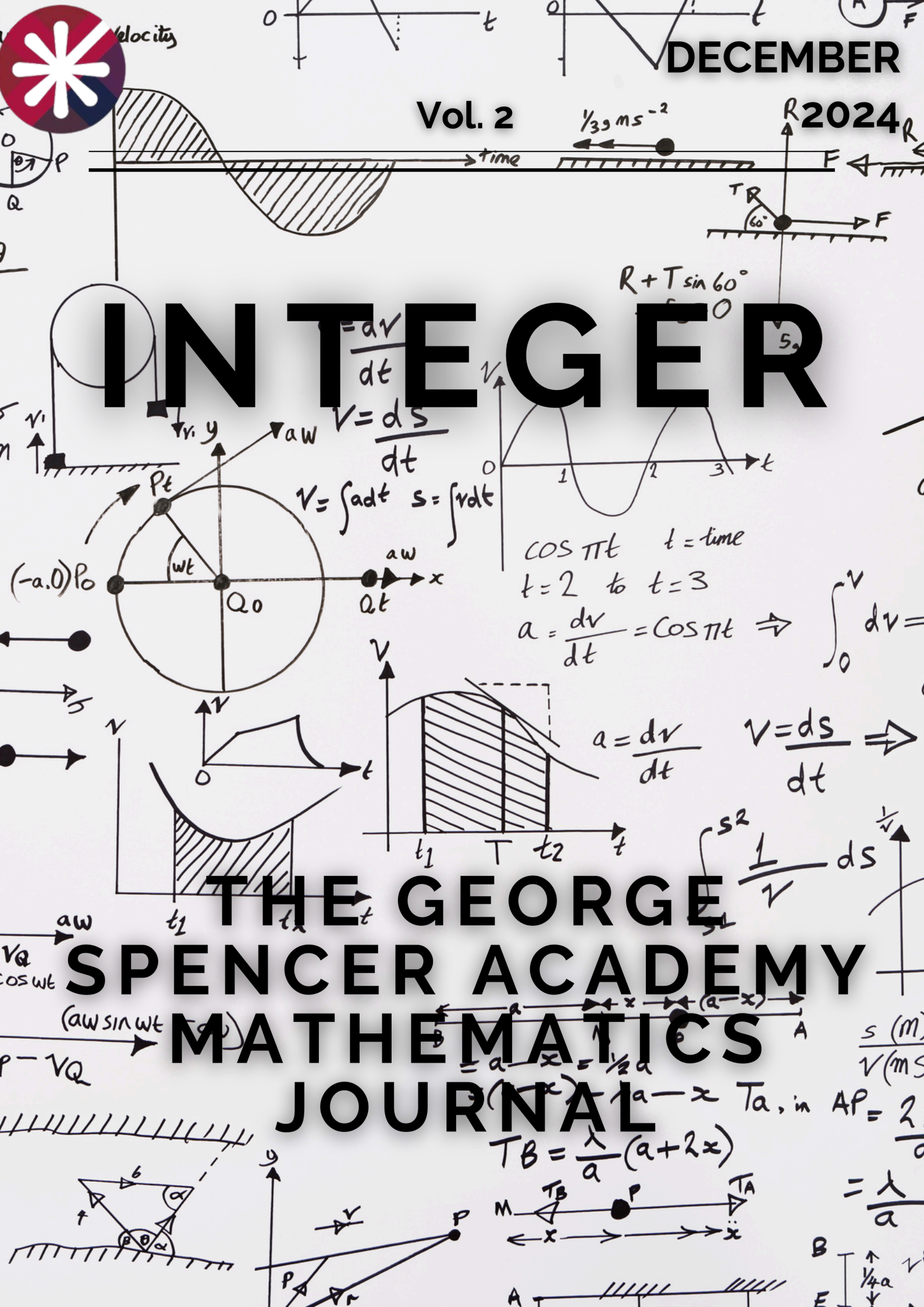


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INTEGER

THE GEORGE SPENCER ACADEMY MATHEMATICS JOURNAL

"The second edition of Integer goes to show how much talent we have within our Year 12 cohort this academic year. With such a wide variety of articles covering a fascinating range of topics, this edition is essential reading for anyone preparing to study Mathematics at A Level and beyond."

Miss J. Cullen
Head of Key Stage 5 Mathematics

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Is 37 special?



Alex Cullen

IS 37 SPECIAL?

Alex Cullen

Introduction

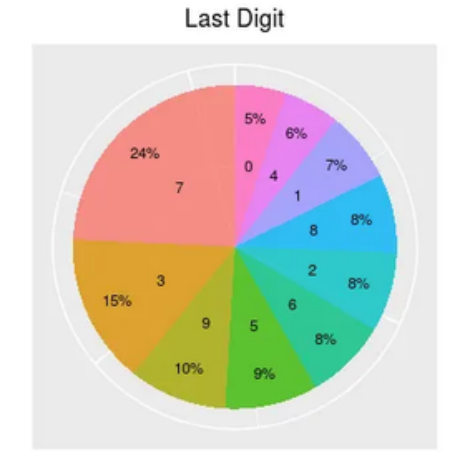
The number 37 is the 12th prime number (an integer divisible only by 1 and itself). It is the atomic number of rubidium and is the 'normal' temperature of the human body in degrees Celsius. Shakespeare wrote 37 plays, and Jesus performed 37 miracles in the Bible. A Tyrannosaurus Rex has 37 vertebrae in its tail, and $37/3.7 = 3^{+7}$.

The number 37 seems to appear frequently, in the places you would least expect. This is, most likely, a mere coincidence. Shakespeare had no reason to stop at 37 plays, and the human body did not evolve to function at 37°c for any reason other than enzyme function.

And yet he did, and yet it does. So is 37 special, or are these just coincidences?

Perceived randomness

The blue-seven phenomenon



When picking a number between 1 and 100, respondents pick a number ending in 7 around 24% of the time (Source: Akio Nakamura)

Psychologists coined the term 'blue-seven phenomenon' after a study was carried out in the 20th century. It found that when asked to pick a 'random' colour and number between 1-10, participants reliably picked 'blue' and 'seven'. This trend comfortably fits the results of many different cultures and locations.

How does 37 fit perceived randomness?

A study conducted by Veritasium in 2024, found that out of over 200,000 responses, there were some surprising consistencies, despite participants being required to be random.
The survey consisted of two parts:

- 'Enter a random number between 1-100';
- 'Enter the number you think the fewest people submitted'.

The results of the first part were clear. Once discounting the outliers of 1, 2, 99 and 100 (as people were primed in the question), and once discounting the culturally relevant numbers 69 and 42, four obvious numbers stood out: 7, 73, 77 and 37.

Interestingly, the results to the second question followed a similar pattern. Ignoring the extremities and non-random numbers, two numbers stood out: 37 and 73. This is particularly interesting, as we see that even when specifically asked to pick the least picked number, people actually picked two of the most picked numbers.

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But why 37 and 73?

Firstly, people tend to perceive odd numbers as "more random" than even numbers. In addition, any multiples of five are also seen as "less random" than non-multiples of five. More specifically, people perceive prime numbers as more random than composite numbers. This could be because prime numbers are far less prominent in our daily lives. The number of pixels on a screen is always a composite number. The interest rate of several years; our means of keeping time; the number of centimetres in a metre or kilograms in a ton; the number of hours in a day and days in a year: wherever you look, composite numbers thrive.

However, this still doesn't explain why 37 and 73 are 'more random' than 53 or 71 for example. To understand this, we can simplify the interval to 1-10. 1 and 10 aren't perceived as random because they are there, already in the interval. 2, 4, 6 and 8 can be discounted because they are even. And 5 can be discounted because it is both the halfway mark, and a multiple of 5. Finally, 9 is a multiple of 3 so can be discarded. This leaves us with two numbers: 3 and 7.

If we state that 37 and 73 are the numbers with the highest level of perceived randomness, then it makes sense that we observe it more in our daily lives. We aren't expecting to find these two primes in our lives, so when we do, it comes as a surprise. This leads to confirmation bias. People will notice 37 and 73 and therefore will subconsciously look out for them more. So they see them more frequently and this leads to a cycle.

The median value for the second prime factor of an integer

The smallest prime factor

This sounds complicated but it can be simplified.

Consider a composite number **n**. The probability that the smallest prime factor of **n** is 2 is 0.5, as the smallest prime factor possible is 2 and half of all integers are divisible by 2.

Now the probability of the smallest prime factor of **n** being 3 must be calculated differently, as half of all multiples of 3 are also multiples of 2 and so 3 would not be the smallest prime factor. This means we must multiply ⅓ by ½ to lead to the probability of **n** having 3 as its smallest prime factor being ⅙. We can continue to calculate the probability of each prime being the smallest prime factor of a number, but we have already reached the median value: 2, since half of all numbers are divisible by 2.

The second smallest prime factor

First, 2 cannot be the second smallest prime factor because it is the smallest prime number. So we start with 3. The probability that n is divisible by both 2 and 3 is ½ x ⅓, or ⅙. The probability that a number's second smallest prime factor is 5 is possible on 2 conditions. It will either be divisible by 2 and 5 or 3 and 5, but not 2, 3 and 5, since that would make 3 the second smallest prime factor. First, if n is divisible by 2 and 5 but not 3. ⅕ x ½ gives 1/10.

However, this does include multiples of 3 as well. For this reason, we must multiply

by ⅔ to give 1/15. Second, if n is divisible by 3 and 5 but not 2. ⅓ x ⅕ gives 1/15.

But this does not exclude multiples of 2, so by halving to get 1/30, we find the probability that a number is divisible by 3 and 5.

Finally, by adding these two probabilities together, we can find the overall probability that a number had 5 as its second smallest prime factor. 1/15 + 1/30 gives 1/10. Therefore, the probability that a number has five as its second smallest prime factor is 1/10.

Again, we can continue to calculate the probabilities of each prime being the second smallest prime factor of a number n.

When we add all these probabilities together, something interesting happens when we reach 50%. **37 is in the exact middle.** This means that half of all numbers have a second smallest prime factor that is 37 or less. 37 appears again.

Conclusion

Overall, it is fair to say that the number 37 crops up a lot: whether it be in geometry, science, maths or culture. It seems to make its way everywhere.

It has some very useful properties, being the median value for the second smallest prime factor of all integers, and being one of the numbers with the highest levels of perceived randomness. The 12th prime number is 37, and the 21st prime number is 73.

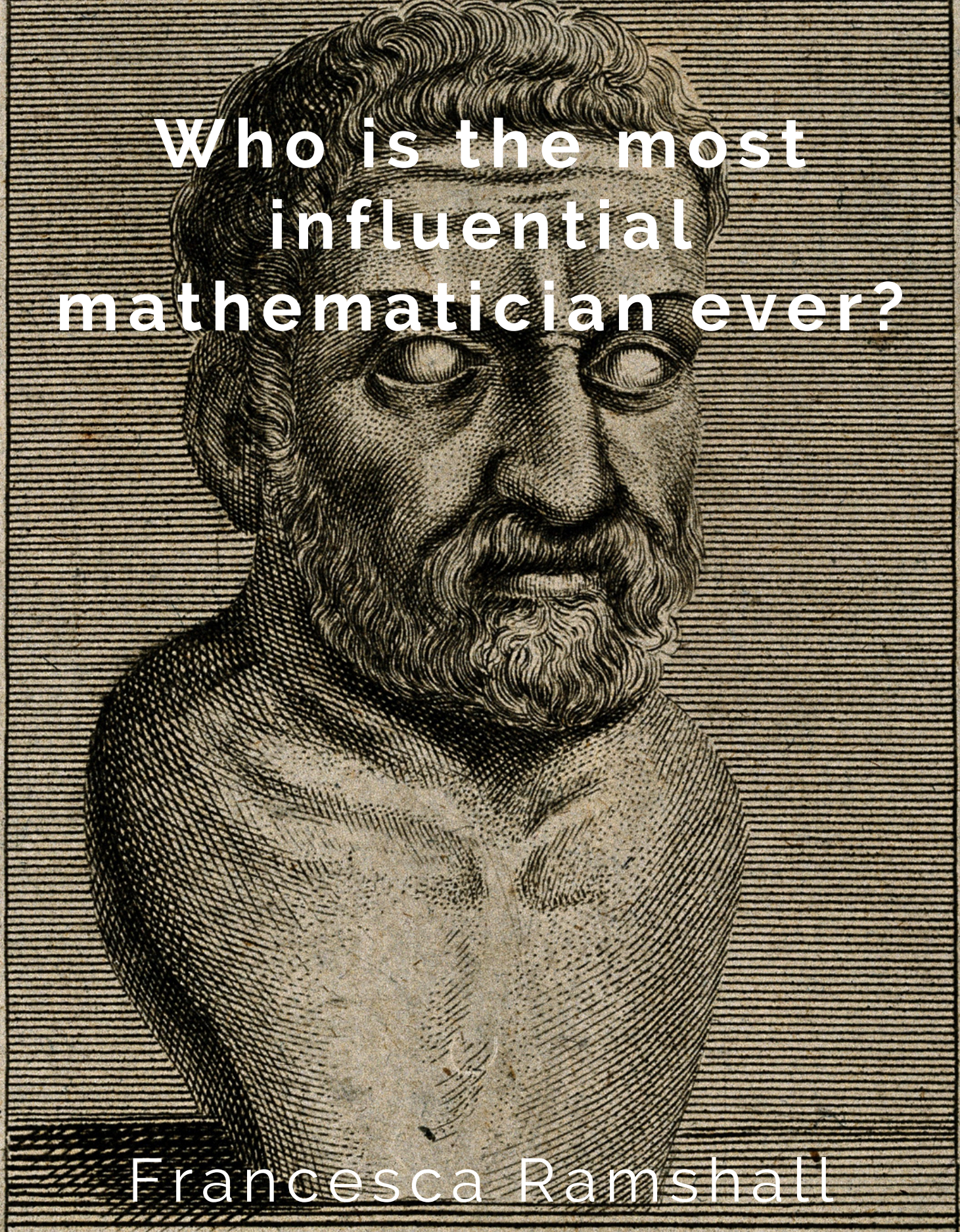
37 is certainly a fascinating number, but is it special? Probably not. All numbers have their strengths.



"HALF OF ALL NUMBERS HAVE A SECOND SMALLEST PRIME FACTOR THAT IS 37 OR LESS"

Alex Cullen

Who is the most influential mathematician ever?



Francesca Ramshall



WHO IS THE MOST INFLUENTIAL MATHEMATICIAN EVER?

Francesca Ramshall

Introduction

Who is the most influential mathematician ever? This, no doubt, is a huge and relatively subjective question. Perhaps it is even impossible to answer?

However, we can investigate some of the most influential mathematicians to try and paint a picture of how modern maths came to be. From Blaise Pascal to Pythagoras, from Isaac Newton to Ada Lovelace, from Archimedes to Alan Turing: all have contributed significantly to the shaping of the discipline, but is one more influential than the rest?

To try to answer this question, I will be judging the chosen mathematicians on three main criteria:

- How many discoveries/contributions did they make?
- Were their discoveries correct?
- How often are their discoveries used in real life, and how can they be used in practical ways?



Thales of Miletus (source: Creative Commons)

Naturally, I will investigate these mathematicians in chronological order to fully understand the roles they played in shaping the field of mathematics.

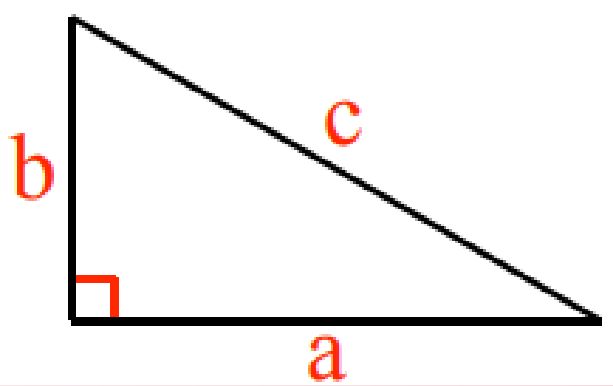
Thales of Miletus (c.624-c.546 BCE)

Thales of Miletus is said to be the first true mathematician and first known mathematician to be attributed to a mathematical discovery. This, in itself, makes him a hugely significant figure.

He is credited with the first use of deductive reasoning applied to geometry by deriving four corollaries to Thales' theorem and he predicted a solar eclipse. When we assess this against our criteria, Thales' discovery has been proved to be correct and is critical in modern engineering and architecture to ensure that certain proportions are maintained, which is essential for the stability and aesthetics of structures like buildings, viaducts and bridges. Moreover, the theorem can be applied in everyday situations, such as measuring inaccessible heights using shadows. However, this theorem is his main discovery and therefore he didn't really contribute as much to mathematics as others, and so cannot be called the most influential mathematician ever.

Pythagoras of Samos (c.570-c.490 BCE)

There is no doubt that Pythagoras is an influential mathematician, as his theorem is still commonly used and taught in schools. It states "the sum of the areas of the two squares on the legs equals the area of the square on the hypotenuse", or "a squared + b squared = c squared".



Pythagoras is also credited with the geometric construction of the first perfect solids and the discovery of perfect and amicable numbers, as well as polygonal numbers. His work with triangles and the square root is a cornerstone of the discipline. This theorem has also been proven correct and is used for many practical applications, such as the development of engineering, oceanography, calculating trigonometric ratios and even road transportation and putting up pylons.

However, because he lived so long ago, it is unclear how many of his discoveries can truly be attributed to him and how many were developed later by the Pythagorean School. Nevertheless, Pythagoras is definitely a contender for the title of most influential mathematician ever.

Euclid (c.330-c.270 BCE)

Euclid was an ancient Greek mathematician active as a geometer and logician. He is often hailed as the father of geometry and is mainly known for the **Elements Treatise**, which established the foundations of geometry that largely dominated the field until the early 19th century. This is now known as Euclidean Geometry.

While much of the content in his book was from previous mathematicians, he added many proofs, and **Elements** is the oldest large-scale deductive treatment of mathematics. Furthermore, Euclidean geometry is still as valid today as it was 2,300 years ago. It is widely used in many disciplines, including art, architecture, science and engineering, to name but a few. Therefore, many commentators cite him as one of the most influential figures in the history of mathematics.



A double page from Euclid's **Elements**, published 1482 (Source: Creative Commons)

Archimedes (c.287-c.212 BCE)

Archimedes, known as the "Father of Mathematics", invented the crucial sciences of mechanics and hydrostatics. He created one of the most essential concepts in Physics: the centre of mass. He also discovered the laws of levers and pulleys that allow moving heavy objects using small forces.

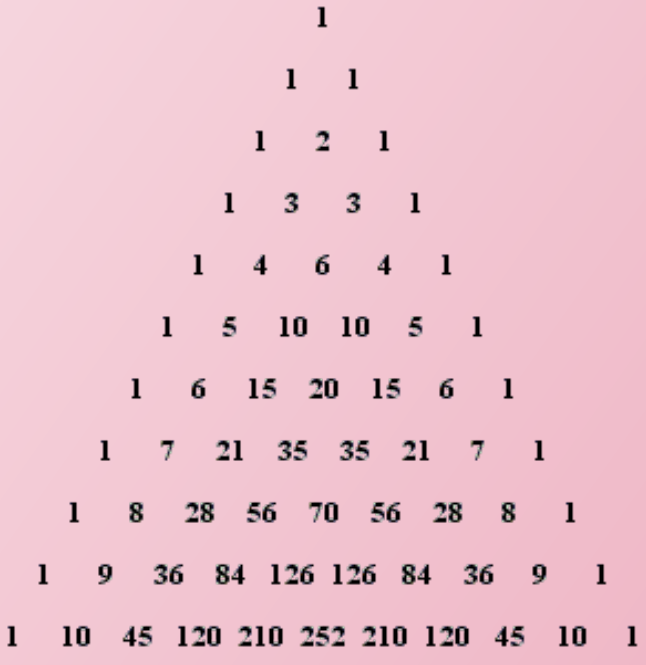
Furthermore, Archimedes calculated the most precise value of pi. The fraction $22/7$ was his upper limit of pi, and this value is still used practically today.

Archimedes also discovered mathematically verified formulas for the volume and surface area of a sphere and he also showed how exponents could be used to write more significant numbers. The significance of these discoveries, as well as the wide-ranging nature of them, arguably makes Archimedes the most influential ancient mathematician. He seems to have made more relevant and practical contributions than those who came before him, and his discoveries are crucial to the foundations of modern mathematics.

Blaise Pascal (1623-1662)

Pascal created one of the first digital calculators in 1645, called the Pascaline. It used dials, gears and pins to perform multi-digit addition, subtraction, multiplication and division operations. In addition, Pascal made many mathematical discoveries, including Pascal's Theorem, Pascal's Triangle (used in binomial expansion) and his contributions to probability theory. While Pascal's Triangle is commonly used in probability theory, combinatorics and algebra, it perhaps has less of a practical use compared to other discoveries. While some of his scientific inventions are incredibly influential, his contributions to mathematics, while crucial, are perhaps not as influential as others on this list.

Furthermore, it should be noted that mathematicians from Pascal onwards have been influenced heavily by those aforementioned. For example, Pascal learned geometry from Euclid's **Elements**. Does this make the ancient mathematicians more influential? Or are more recent mathematicians more influential, due to the increased modern applications of their work?



Pascal's Triangle (Source: Creative Commons)

Isaac Newton (1643-1727)

As well as his major contributions to the field of physics, Newton is most often credited with inventing calculus (despite the Newton-Leibniz calculus controversy), which is the mathematical study of continuous change. He also generalised the binomial theorem to any real number, contributed to the study of power series, developed a method for approximating the roots of a function, classified most of the cubic plane curves, and also originated the Newton-Cotes formulas for numerical integration.

There is no doubt that Newton was a remarkable mathematician and played a significant role in the development of mathematics as well as physics and chemistry. But was he the most influential, or are his contributions to science more significant?

Leonhard Euler (1707-1783)

Perhaps the best-known of Euler's findings is the Euler identity (covered in two of the other articles in this edition). The Euler identity is:

$e^{i\pi} + 1 = 0$

Often called the most beautiful equation in mathematics, this formula shows the relation between five mathematical constants: e, i, π, 1, and 0. It has wide-ranging uses in mathematics and physics, including in 21st-century electronics.

Euler also created the mathematical function notation, f(x), where f stands for "function" and (x) the variable of the function. This notation is used widely today.

Furthermore, Euler is said to have founded the studies of graph theory and topology and made pioneering discoveries in other branches of mathematics, such as analytic number theory, complex analysis and infinitesimal calculus. For these reasons Euler is regarded as one of the greatest, most prolific mathematicians in history and the greatest of the 18th century.

"EULER IS REGARDED AS ONE OF THE GREATEST, MOST PROLIFIC MATHEMATICIANS IN HISTORY"

Francesca Ramshall

Carl Friedrich Gauss (1815-1852)

Gauss's mathematical genius was first apparent when, at the age of three, he detected an error in his father's computations, despite never having been taught arithmetic. As part of his doctoral dissertation (at the age of 21), Gauss was one of the first to prove the fundamental theorem of algebra and later on he was credited with being the person to first discover and study non-Euclidean geometry, even coining the term.

Gauss also worked out the construction of a 17-sided figure with only a compass and ruler. He deeply influenced the development of many branches of mathematics, such as algebra and differential geometry, and initiated the theory of numbers.



Carl Friedrich Gauss (Source: public domain)

Gauss is commonly described as one of the greatest mathematicians of all time, and he did have massive influence through his discovery of non-Euclidean geometry. However, he did not publish all of his discoveries, and so his greatness cannot be understood fully.

Ada Lovelace (1815-1852)

Lovelace is the first woman in our list, as female mathematicians were not recorded as effectively as their male counterparts throughout history. However, there is no doubt that Lovelace made massive contributions to mathematics and is known to be the first computer programmer. She worked with Charles Babbage on an advanced machine called the Analytical Engine. In 1843, Lovelace translated a French paper that Italian mathematician Luigi Menabrea wrote regarding the Analytical Engine. She also added thousands of words of her own notes to the paper. Lovelace realised that the Analytical Engine could carry out an extensive sequence of mathematical operations. The example she wrote of one such sequence - how to calculate Bernoulli numbers - is regarded by computer historians as the first computer program. She even speculated that the Analytical Engine could be used to perform operations on "other things besides number", such as musical notes.

While the Analytical Engine was never completed, Lovelace's discoveries and predictions shaped the future of mathematical computer science. But her first-hand contributions to mathematics are perhaps less influential than others on this list.

Alan Turing (1912-1954)

Alan Turing made significant contributions to mathematics, including proving that there cannot exist any universal algorithmic method of determining truth in mathematics, introducing the concept of the Turing machine (which laid the foundation for modern computers), and demonstrating that logical systems are undecidable.

In the Second World War, Turing was a pioneer in breaking German ciphers at Bletchley Park in England, where he made major advances in deciphering messages encrypted by the German "Enigma" cipher machine.

Turing's contributions changed modern computing forever; without his discoveries, the future of both the computer and history could have been different.



A German Enigma machine (credit: Flickr)

Conclusion

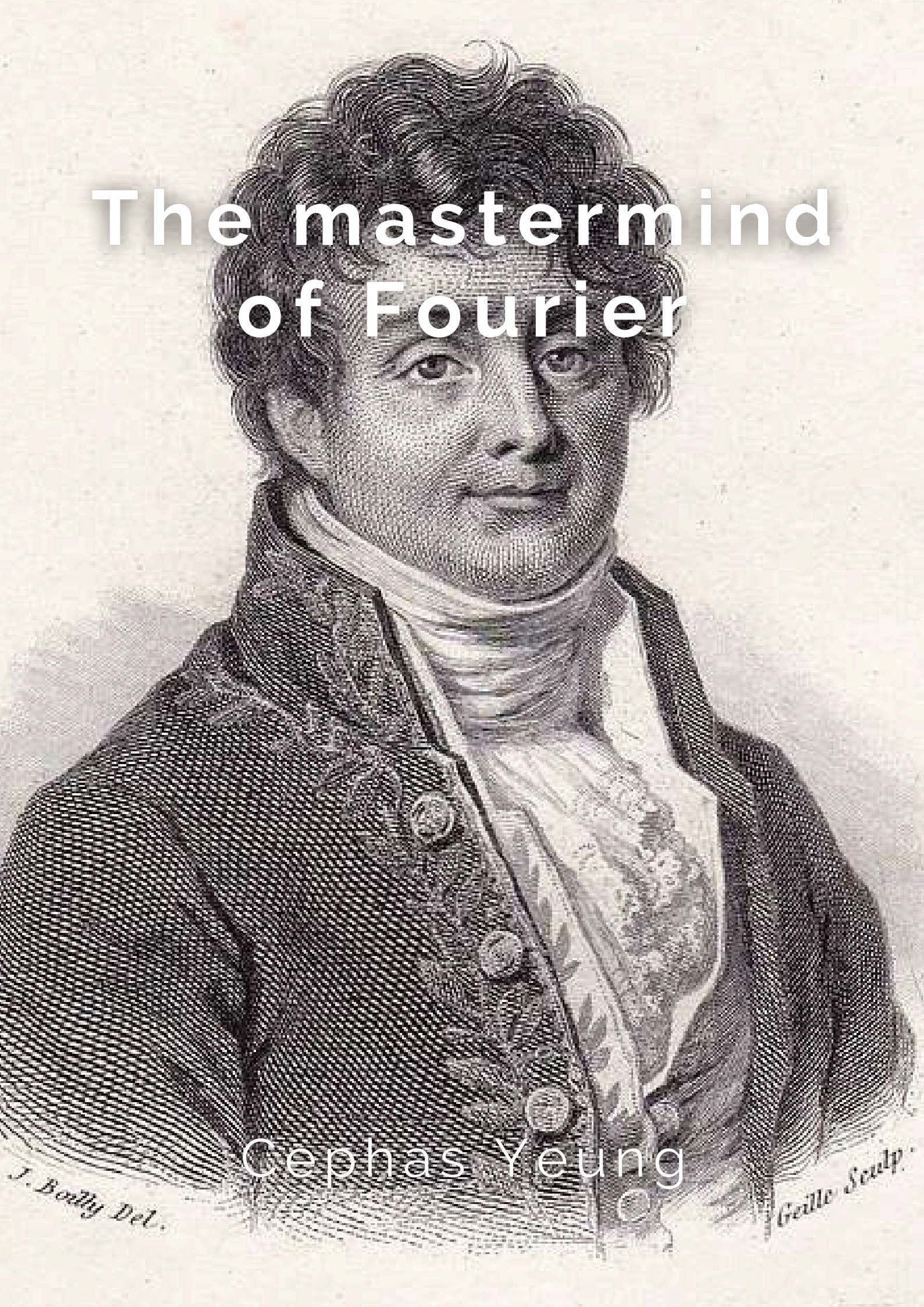
All of these mathematicians contributed massively to the world of mathematics, each influencing the field in a different way. This list is not extensive, and some honorary mentions include:

- Hypatia (c.350-415 CE), one of the first recorded female mathematicians before she was murdered by a mob of Christians;
- Srinivasa Ramanujan (1887-1920), an Indian mathematician who was regarded as a genius in his field despite lacking access to formal training in pure mathematics;
- Gottfried Wilhelm Leibniz (1646-1716), also credited with inventing calculus at the same time as Newton.

With regard to the title question, using the initial criteria, I think the most likely winner of the title 'most influential mathematician' is Pythagoras. His mathematical findings have influenced future discoveries, and his contributions have been proven to be correct countless times - remarkable considering he discovered them thousands of years ago without any modern equipment. His theorem is still used in schools and has numerous practical applications today. Finally, he set up the Pythagorean School in Croton (modern-day Italy) where he taught his mathematical findings, amongst other topics, to students.

However, there is no definitive way to answer the question; as I have said previously, it is incredibly subjective. So, what do you think? Who deserves the title of the most influential mathematician ever?

The mastermind of Fourier



Cephas Yeung

THE MASTERMIND OF FOURIER

Cephas Yeung

Introduction

There are many famous mathematicians such as Albert Einstein, Alan Turing, Isaac Newton and Pythagoras. But you may not have heard of a French mathematician called Jean-Baptiste-Joseph Fourier, even though his past contributions sparked global changes in the fields of mathematics and physics, including the discovery of the greenhouse effect.

A lot of modern-day technology also relies on the work of Fourier, including images and videos on the Internet - image compression algorithms you are using right now. If you are downloading this journal via the Internet, you are using digital signal transfer via WiFi or mobile cellular data, both influenced by Fourier's work. Not only that, but Fourier's work nearly stopped the nuclear arms race during the Cold War! He is arguably the greatest mathematician of all time.

Biography

Fourier was born in Auxerre (southeast of Paris) in 1768. He was orphaned at the age of 9. Fourier was first educated at Pallais's school, run by a music master from the nearby cathedral. He first succeeded in Latin and French. Later on, he would proceed to the École Royale Militaire of Auxerre where at the age of thirteen, mathematics became his real interest.

Fourier later joined Napoleon's army in the invasion of Egypt as scientific adviser, where he helped contribute several mathematical papers to the Egyptian Institute. After Britain defeated France here, Fourier returned to France where he was appointed "préfet" (French term for Governor) of the department of Isère in Grenoble by Napoleon. During his time in Grenoble, he began experimenting with propagation of heat.

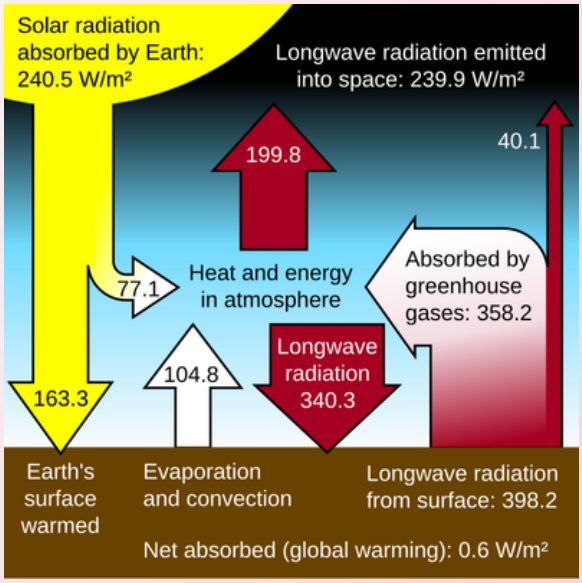
Fourier's work enabled him to express the conduction of heat in two-dimensional objects in terms of differential equations, which led to the discovery of the law of heat conduction: Fourier's Law. Today, his name has been written on the side of the Eiffel Tower to commemorate his mathematical and science contributions.

Fourier and the Greenhouse Effect

In 1827, Fourier questioned the reason why the Earth's average temperature was believed to be approximately 15˚C. He hypothesised that there must be some type of balance between the incoming and outgoing energy to maintain the balance. His calculations indicated that the Earth should actually be colder.

To have the average temperature of 15˚C, Fourier theorised that there must be a process occurring in the atmosphere. This led to the development of knowledge of how the atmosphere works, like a

greenhouse, where the greenhouse shell absorbs the heat and some is emitted back. This led to the development of the understanding of global warming. After more than 150 years since Fourier's discovery of the greenhouse effect, we only truly realised the effect of industrial development on the environment.



The mathematics of the greenhouse effect (Source: Creative Commons)

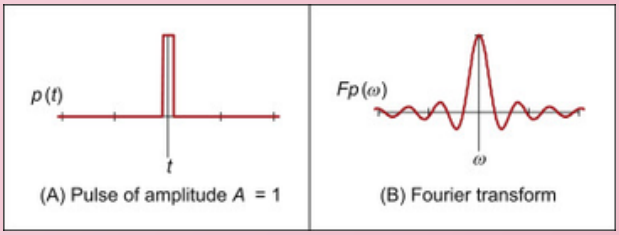
Development of the Fourier Transform algorithm

You may never have heard of the idea of Fourier Transform, but in our day-to-day life, Fourier Transform is being utilised all around us in the ever-changing digital world. The basic premise of Fourier Transform is to allow the conversion of signals from its frequency and time (called the time domain), to the corresponding amplitude and frequency (called the frequency domain).

However, Fourier Transform is only used when data is perfectly continuous, like a sine wave. As time goes on, new developments into different types of Fourier Transform emerge, such as the Discrete Fourier Transform where, as the name suggests, discrete data is used for real-life situations like sound recording, where there is a finite number of data points.

A great example of this in practice is converting a sound file with an annoyingly high pitch created by disturbance. Firstly, by using the Fourier Transform algorithm, the sound is converted into just pitch and its corresponding amplitudes. Then, the annoying pitch is removed. Finally, the sound is converted back using the inverse Fourier Transform into a sound file. In a digital world, where everything is made from 0s and 1s, there are a lot of signals that require processing, such as every audio signal device. Thus, Fourier Transform is essential in signal processing.

Fourier Transform will become useful in the future of different areas of maths, physics and computer science, such as the development of quantum computing.



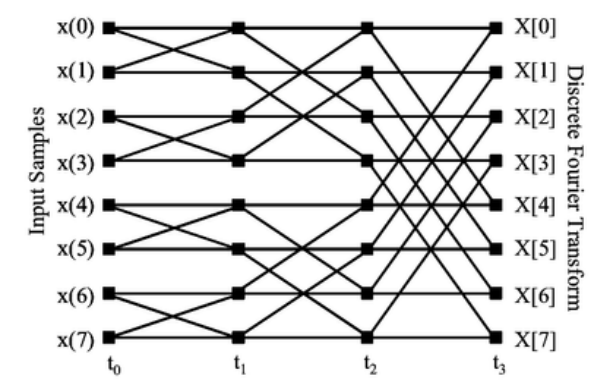
An overview of Fourier Transform (Source: ScienceDirect)

How Fourier nearly stopped the nuclear arms race

The Fourier Transform has been utilised further in trying to stop nuclear weapons from being deployed. In 1963, the Limited Test Ban Treaty was signed by the United States, the Soviet Union and Great Britain. The treaty was designed to prohibit nuclear weapons tests or other nuclear explosions underwater, in the atmosphere or in outer space. But it was not banned for nuclear testing underground because it is simply impossible to detect nuclear testing underground. If you placed a seismometer underground, it is very hard to differentiate between normal earth seismic activity and nuclear testing-induced seismic activity.

John Tukey and James Cooley developed a new type of Discrete Fourier Transform called the Fast Fourier Transform (FFT). As the name suggests, FFT increases the speed of the algorithm compared to previous iterations.

The research paper was published by Tukey and Cooley in 1965, which was too late to bring about a full ban on nuclear testing, as other nations had joined the USA and the Soviet Union in becoming major nuclear powers. If Tukey and Cooley had published the paper earlier, perhaps there would be a more comprehensive ban on the uses and testing of nuclear weapons.



A flow diagram for the Cooley-Tukey FFT. (Source: Gregory Donohoe)

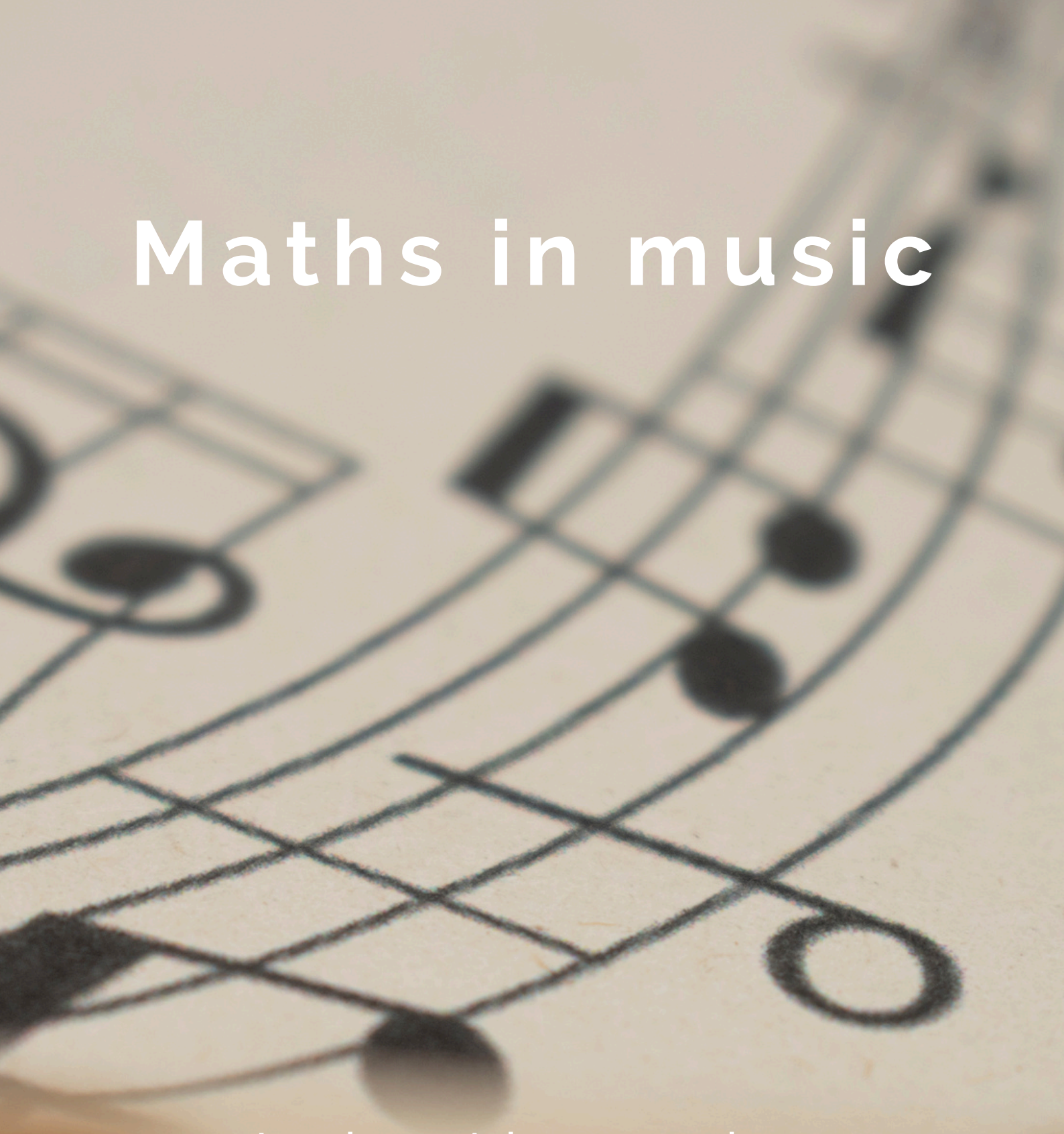
Conclusion

The reason why I researched Fourier is because his creations led to a better and potentially more peaceful world. His fascinating inventions and discoveries have helped to tackle mankind's problems in the 20th and 21st centuries.

"FOURIER TRANSFORM WILL BECOME USEFUL IN THE FUTURE OF F MATHS, PHYSICS AND COMPUTER SCIENCE"

Cephas Yeung

Maths in music



Luke Alexander

MATHS IN MUSIC

Luke Alexander

Introduction

Music is everywhere. The presence of music is felt across the entire globe, and it's beautiful. From death metal blasted through a 1000W speaker, to a heart-breaking performance of Beethoven's "Moonlight Sonata", to heavenly birdsong, all of us have experienced music.

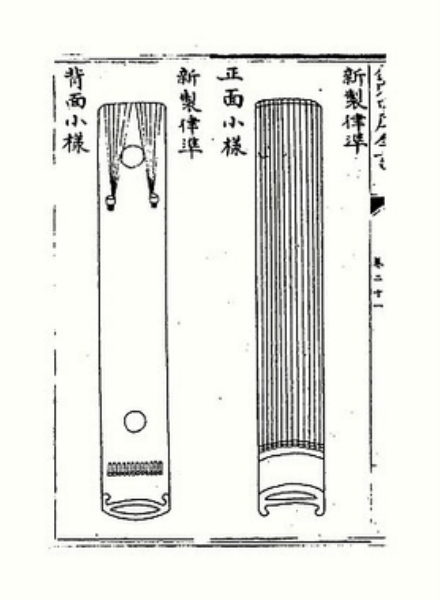
The importance of maths in music cannot be understated. Maths connects music. It allows us to define and standardise a system which can be used by any musician, with any instrument, anywhere. Using maths, we can explain harmony and develop music theory, allowing for a common standard of music to be taught and be learned exceptionally easily. Essentially, maths has allowed us to categorise how we can make music sound "good".

This, arguably, has led to the very large majority of music most of us listen to today. If we go back to the very basics, our mathematical system actually defines the standard 12-note scale. Without this, the music we listen to would be completely different.

The Western Standard

Almost all tuning systems are derived from the octave - the interval between one note and another which is double or half its frequency. For example, orchestras tune to the note A (frequency 440 Hz), but the note with frequency 880 Hz is also A, just an octave higher, and the note with frequency 220 Hz is A, but an octave lower.

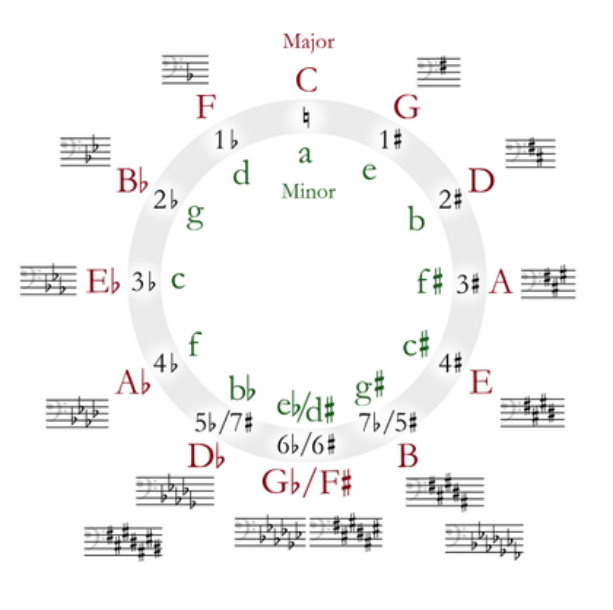
The most common tuning system today, and the standard in the West, divides the octave into twelve intervals of equal size. This means the frequency ratio between two adjacent intervals will be the 12th root of 2, approximately 1.059463. Using this, absolute values for notes can be calculated and used for tuning any instrument. This is done by calculating every note relative to a predefined note (A=440 Hz). This tuning system is known as 12 TET, or 12 Tone Equal Temperament tuning.



The 12-string equal temperament tuning instrument devised by Zhu Zaiyu in 16th-century China (Source: Creative Commons)

This allows us to see clearly how maths has defined the standard musical system that we all know, since everything in Western music is based on these 12 notes. It leads to common ideas in theory such as the circle of fifths, intervals, chords, scales and keys - or in other words, the study of harmony. Harmony is the concept behind combining sounds together in order to create new musical ideas (essentially, the study of how we group notes together to make them sound "good").

For example, a basic three-note chord is made from using the first (root), third and fifth note in a major or minor key. These chords can then be ordered using cadences (a method of categorising how "resolute" a phrase feels) which leads to the creation of songs.



The Circle of Fifths (Source: Creative Commons)

However, even though we can see that music is based heavily on a mathematical system, it's important to recognise that music, even if it originates from the same 12-note tuning system, doesn't exist because of maths and basic definition. It exists because of artists who are able to not only manipulate and work with a system like this, but also move away from it and work via intuition. Maths does define a standard in music, but it isn't - and never will be - a constraint for creativity. For example, artists may experiment with microtonal music (splitting the octave into more or less than 12 equal intervals) like Aphex Twin, or bands like Radiohead moving away from standards by writing songs in irregular time signatures and experimenting with electronic music (albums like Kid A). Yet defining music mathematically also paves the way for unlimited creative expression - if you understand exactly what notes work together, and why, then it opens the door to incredibly technical improvisation.

Issues with the Western Standard

Although the western standard works well enough for general use, there are slight discrepancies. A good example of this is the Pythagorean comma (the distance between two enharmonically equivalent notes, for example B# and C). It can be derived from determining notes using the frequency ratio from a perfect fifth (3:2) and the octave (2:1) by multiplying a given pitch by these frequencies. The Pythagorean comma (~1 Hz) between enharmonic equivalents means that pianos can never be perfectly in tune. Although most people cannot notice the difference, those with perfect pitch are able to tell.

Non-standard musical systems

There are many other commonly used tuning systems which differ from the 12 TET Western standard. However, most of these are still standardised using ratios to split up the octave into equal intervals.

There are microtonal systems using 19 TET, 31 TET (such as Zheanna Erose) and even 53 TET. Having such a dynamic range of notes leads to lots of interesting theory and harmony.

Music from non-western cultures often uses different tuning systems. For example, Indian classical music splits the octave into twelve parts, similarly to western music, but it uses a just-intonation system instead of an equal-temperament system. This means it uses exact frequency ratios to determine each note, and so each tuning is considered pure: exactly in tune. This avoids the problem of the Pythagorean comma.

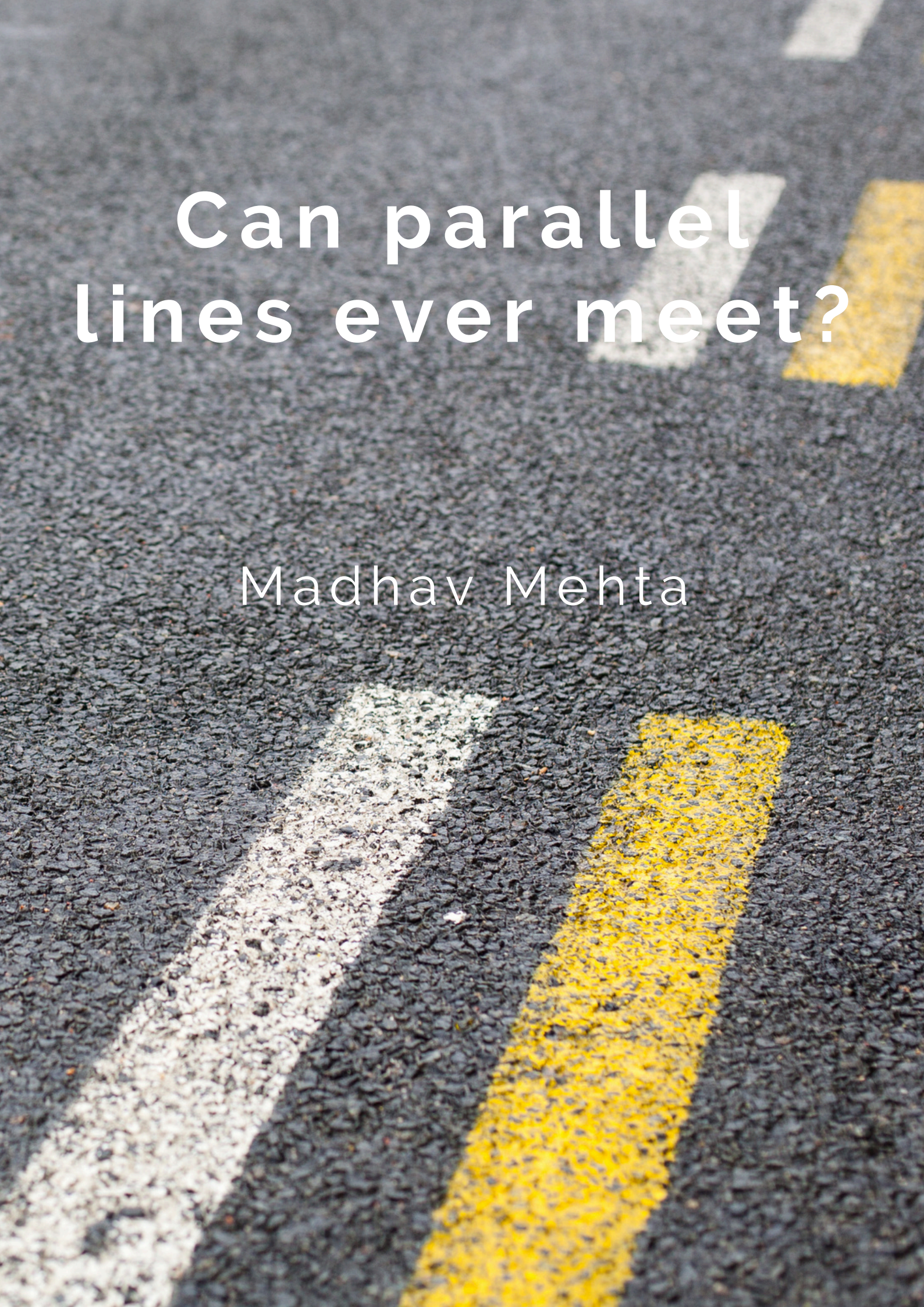
Conclusion

Although it can be argued that a lot of music is driven by creativity and intuitive expression, it is undeniable that music is underpinned by mathematical structure. More often than not, it is this structure that gives us the freedom to create and express ourselves through music. Imagine how much harder it would be for musicians if we had to learn to create music with an essentially infinite choice of notes, with no idea how to construct chords, scales or patterns like the circle of fifths.

Some would thrive with this sort of limitless freedom, but most would struggle, and music would be unpleasantly irregular, hard to teach and share. Without maths, perhaps music would have had less of an influence on us all.

“IT IS UNDENIABLE THAT MUSIC IS UNDERPINNED BY MATHEMATICAL STRUCTURE... THAT GIVES US THE FREEDOM TO CREATE AND EXPRESS OURSELVES THROUGH MUSIC”

Luke Alexander



Can parallel lines ever meet?

Madhav Mehta

CAN PARALLEL LINES EVER MEET?

Madhav Mehta

Introduction

What is geometry? Geometry is one of the many branches within mathematics which focuses mainly on the properties of points, shapes, lines/curves and spaces. Over time, geometry has changed from simple concepts to more complex theories.

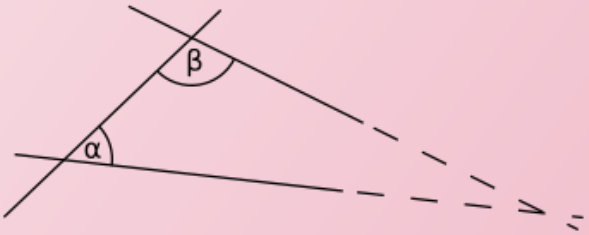
One of the most important parts of geometry is the difference between Euclidean and non-Euclidean geometry. Whilst mathematics students are more familiar than Euclidean geometry (even if they may never have heard of the phrase), non-Euclidean geometry introduces us to concepts that defy traditional beliefs on shapes and spaces.

What is Euclidean geometry?

Euclidean geometry is based on the work of the ancient Greek mathematician Euclid. Euclid was one of the most influential mathematicians of his time and earned the nickname “The Father of Geometry”.

In Euclidean geometry, basic concepts such as points, lines, angles and surfaces are defined with simple axioms and postulates (fundamental statements that are accepted to be true without any proof). These are the building blocks from which all other geometric theories are derived.

One of Euclid’s most famous axioms is the fifth postulate, often referred to as the parallel postulate, which states that if there is a straight line and a point that is not on the straight line, there is only one line that passes through the point and never touches the original line, no matter how far they are both extended. This postulate assumes that the space behaves in a flat, two-dimensional manner, much like the Cartesian plane. This means the parallel postulate essentially justifies the main ideas of flat geometry (for example, that the angles in a triangle always add up to 180 degrees).



Euclidean geometry helps to explain the angles in a triangle adding to 180 degrees (Source: Creative Commons)

Euclidean geometry has been incredibly successful in describing the world around us in many scenarios, as it is the type of geometry that is used in architectural design and engineering. When we think of simple two-dimensional shapes, such as squares, circles and triangles, we all use ideas of Euclidean geometry. However, when we move away from two-dimensional shapes to three dimensions, the type of geometry required changes from Euclidean to non-Euclidean, where the rules are different.

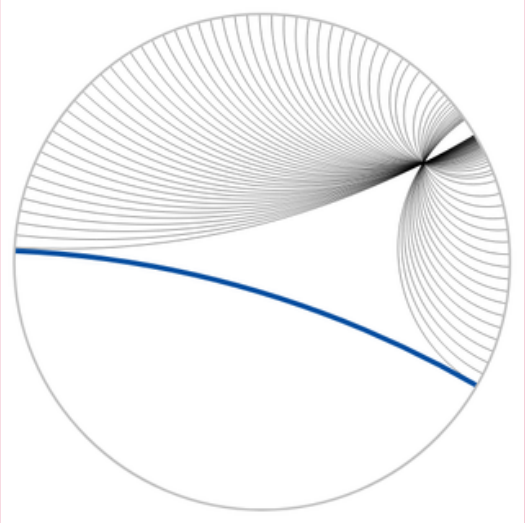
What is non-Euclidean geometry?

Non-Euclidean geometry is a much newer field of geometry created in the 19th century, as mathematicians began to question the validity of Euclid's fifth postulate. While Euclidean geometry assumes that space is flat, non-Euclidean geometry explores what would happen if the space is curved instead. This produces some counterintuitive results, with implications for our understanding of the universe.

Hyperbolic geometry

Hyperbolic geometry is a type of non-Euclidean geometry where space is **negatively curved**, like the shape of a saddle or the surface of a pringle. In this type of geometry, the parallel postulate is replaced by the idea that through any point not given on a line, there are infinitely many parallel lines to the original line. This suggests that the angle of a triangle in hyperbolic space add up to less than 180 degrees, which opposes what we expect in Euclidean geometry (where the angles in a triangle always add up to 180 degrees).

Hyperbolic geometry can be visualised using models such as the Poincaré disk, where lines are represented as arcs of a circle. It has many applications in mathematics, in areas like complex analysis where it can be used to model certain types of functions, and the theory of relativity where it helps to describe the shape of space in the presence of strong gravitational fields.



The Poincaré disk model (Source: Creative Commons)

Elliptic geometry

Elliptic geometry is another type of non-Euclidean geometry and is the opposite of hyperbolic geometry, as space is **positively curved** like the surface of a sphere. This means the parallel postulate is this time replaced by the idea that no two parallel lines can exist, meaning all lines will eventually have to intersect, opposing the nature of parallel lines in hyperbolic geometry.

Elliptic geometry also opposes hyperbolic geometry in many other ways. For example, in elliptic geometry, the angles in a triangle add up to over 180 degrees. The surface of the Earth is a great example to visualise elliptic geometry, because all the lines of longitude will eventually intersect at the two poles.

If we imagine the universe is a curved space, elliptic geometry could help explain why light from distant stars follows a curved path rather than a straight line.

Can parallel lines meet?

We know from Euclidean geometry that parallel lines can never meet. As stated in Euclid's fifth postulate, if there is a straight line and a point that is not on the straight line, there is only one line that passes through the point and never touches the original line, no matter how far they are both extended.

However, in elliptic geometry, there will always be a line that has a point that is not on the line. This would mean it has a parallel line, but from what we know about Euclidean geometry there can be no parallel lines, as they will always intersect. Therefore, parallel lines can never meet even though they fit all the other rules of Euclid's fifth postulate, if the space the lines are in is non-Euclidean.

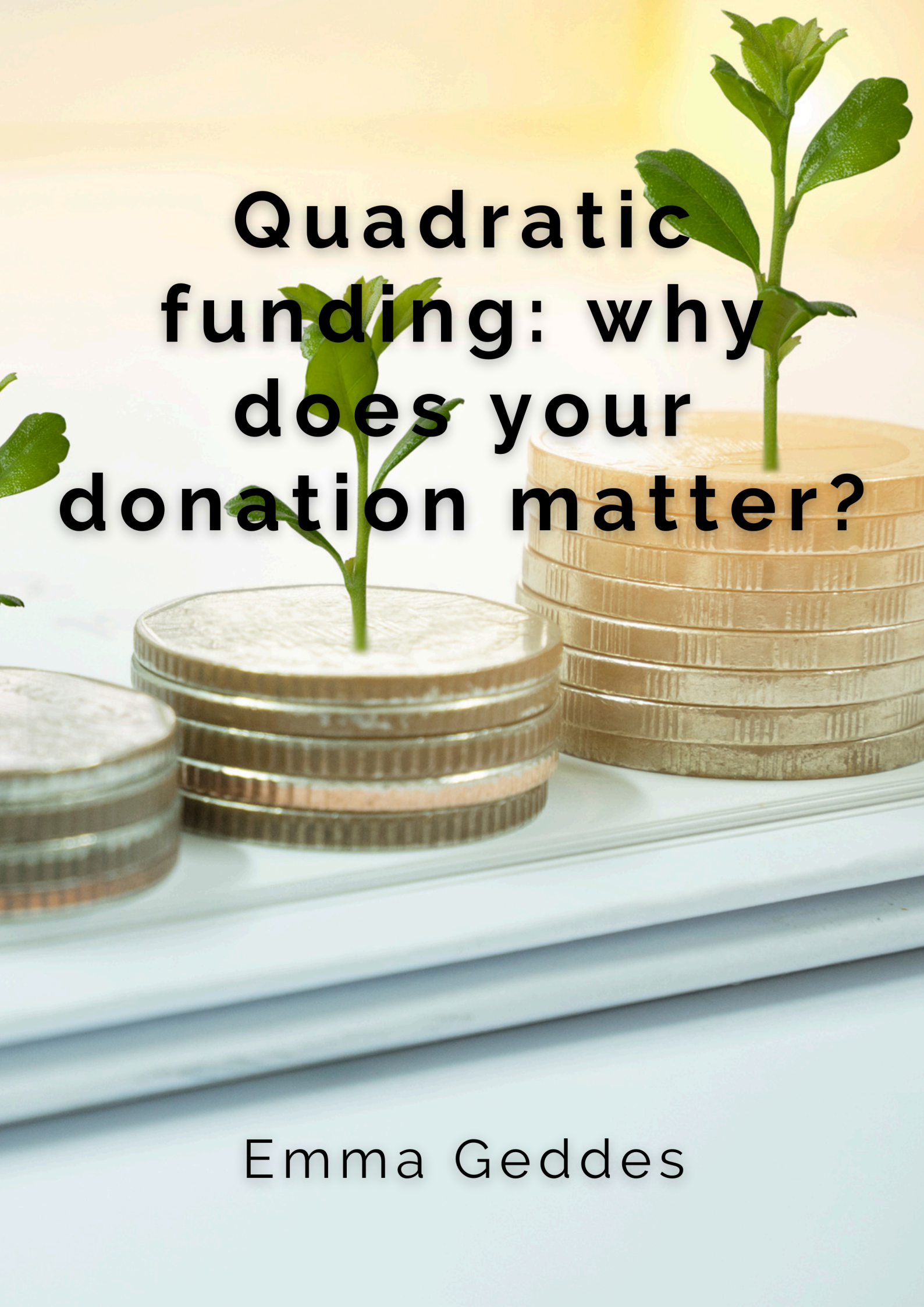
Conclusion

When Euclidean geometry is applied, lines must be on a two-dimensional plane where Euclid's parallel postulate always applies. This means that parallel lines will never meet. However, if the space is changed so that non-Euclidean geometry is applied, there will be either an infinitely large number of parallel lines from just one point that will never touch, or there will be no parallel lines at all, as the lines from the point will touch the original lines.

Therefore, no matter whether Euclidean geometry or non-Euclidean geometry is applied, parallel lines will never meet.

"IN ELLIPTIC GEOMETRY, THERE WILL ALWAYS BE A LINE THAT HAS A POINT THAT IS NOT ON THE LINE... BUT THERE CAN BE NO PARALLEL LINES, AS THEY WILL ALWAYS INTERSECT"

Madhav Mehta



Quadratic funding: why does your donation matter?

Emma Geddes

QUADRATIC FUNDING: WHY DOES YOUR DONATION MATTER?

Emma Geddes

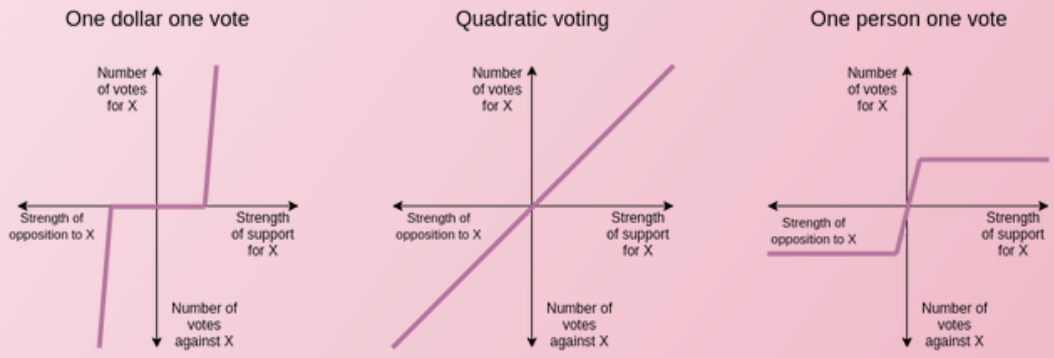
Introduction

Quadratic funding is a form of crowd-funding used for public goods created by Vitalik Buterin (owner of the software Ethereum). Quadratic funding relies on the usage of the quadratic funding formula, in which 'matched amounts' (money donated to public goods from a general matching pool of money given by companies or contributors) are calculated. Public goods are good that are not owned by anyone and can be used by anyone, such as police and fire services, and so there is no profit motive.

The amount received by the project is proportional to the square of the sum of the square roots of the contributions received. This method of funding helps to ensure that the public goods that receive more funding are the ones with the most contributors, rather than the public goods with the most money donated. For example, a project with 100, £1 donations would be funded with more money from the general matching pool than a project with one donation of £100.

Why is quadratic funding good?

Quadratic funding is good for funding public goods as the projects that will get funded the most will be the ones with the most individual donations. This method of funding helps to benefit the general population, and prevents individuals who can donate more than the general public from having a heavy influence over which projects receive the most matched amounts by contributors. Therefore, the majority will be satisfied with the outcomes of this funding.



How quadratic funding/voting compares to "one dollar one vote" and "one person one vote" (Source: Vitalik Buterin)

How does quadratic funding actually work?

The quadratic funding formula is a simple formula in which the amount received by the project is proportional to the square of the sum of the square roots of contributions received.

For example, imagine we had £10,000 in a matching pool and three projects participated in a funding round:

- Project A received £1,000 in funding from five contributors (£200 each);
- Project B also received £1,000, but from two contributors (£500 each);
- Project C received £1,000 as well, but from twenty contributors (£50 each).

Although all three projects received the same amount of money through donations, Project C has had the most contributors, and therefore it will earn the most amount of money from the matching pool.

Project A received an additional £1,851.85, which is 185% of the amount contributed. Project B received just £740.74, which is 74% of the initial amount donated. But Project C received £7,407.41 of matching funding which is 740% of the initial amount that was donated. This is significantly more than the amount from the matching pool given to Project A and Project B.

Conclusion

Quadratic funding is essential to ensure that the funding given to public goods satisfies the majority of contributors, especially those not in the position to donate large amounts of money. Quadratic funding helps to encourage small donations. Often, individuals can be discouraged from donating to projects as they do not see how their relatively meagre £5 can affect the outcome of major projects, but quadratic funding favours the individual contributors over the big donations.

"QUADRATIC FUNDING HELPS TO ENCOURAGE SMALL DONATIONS. OFTEN, INDIVIDUALS CAN BE DISCOURAGED FROM DONATING TO PROJECTS"

Emma Geddes

The Klein bottle



Isaac Potter

THE KLEIN BOTTLE

Isaac Potter

Introduction

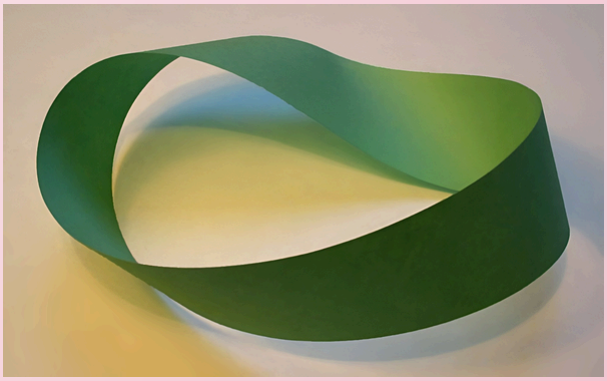
The Klein bottle is arguably one of the most interesting and complex objects in the field of topology (the study of properties of shapes and spaces under continuous deformations). Named after the German mathematician Felix Klein, this so-called "four-dimensional" surface defies our knowledge of geometry and our understanding of dimensions. This is because the bottle simply cannot exist in a three-dimensional space without intersecting itself, it has no "inside" or "outside". So, is the Klein bottle really the first 4-dimensional shape?

What is the Klein bottle?

A Klein bottle is best understood as a non-orientable surface, just like the Möbius strip. The Möbius strip, a one-sided surface, is created by taking a rectangular strip of paper, twisting it 180 degrees and joining its ends. The Klein bottle essentially expands on this idea and brings it into a higher dimension.

Imagine starting with a cylinder, then attempting to bend it and connect its ends to the opposite sides at the same time. It seems impossible. That's because it is. Unlike the Möbius strip, this is impossible to do in our familiar three-dimensional space without the surface "folding in on itself".

Mathematically, the Klein bottle is best described as a shape with no edges or boundaries, meaning it is something known as a "closed surface" and its non-orientability makes it so the concepts of "left" and "right" sides of the shape become almost indistinguishable.



The Möbius strip (Source: Creative Commons)

Constructing a Klein bottle

The challenge of visualising the Klein bottle stems from its reliance on four-dimensional geometry. Practically, a physical model of the Klein bottle must intersect itself, creating an optical illusion rather than a truly accurate representation of the shape. Typically, Klein bottles are modelled as a bottle with its neck passing through its side and merging with the base. However, in a four-dimensional space, the intersection would not occur. This is because there is an additional degree of "freedom" the shape has to avoid crossing itself.

The construction of the Klein bottle is formulated in mathematics using coordinate systems and geometric equations. In abstract terms, it can be described by identifying the opposite edges of a square after a specific twist, similarly to the Möbius strip, but with an additional transformation. Its algebraic representation also highlights its connection to other "four-dimensional" shapes such as the torus ('donut') and projective planes.

Is the Klein bottle really four-dimensional?

On the one hand, the Klein bottle is fundamentally described as a two-dimensional surface. This means that, just like shapes such as spheres, it has only two dimensions of freedom, meaning any point on the shape can be described with just two coordinates. The four-dimensional description is just used to ensure that the shape does not self-intersect, something that is impossible with two coordinates, hence the Klein bottle cannot simply be described as a two-dimensional shape. However, the Klein bottle is not inherently four-dimensional, as its properties are somewhat defined within only 2 coordinates.

On the other hand, the argument for the Klein bottle being four-dimensional is equally compelling. The representation of a Klein bottle being modelled in four dimensions is used to reveal the limitations of three-dimensional spaces. The Klein bottle cannot be two-dimensional or three-dimensional due to its self-intersecting nature, and thus it can only be interpreted as a four-dimensional shape. The dependence of a dimension higher than three dimensions makes the bottle linked to four-dimensional geometry, even if its surface itself remains two-dimensional.

Ultimately, the Klein bottle lies in the middle of these definitions. It exists as a two-dimensional object that cannot fully be described and modelled accurately without the concept of a higher dimensional space. This bottle shows our limited understanding of dimensionality and the boundaries of our spatial perception.

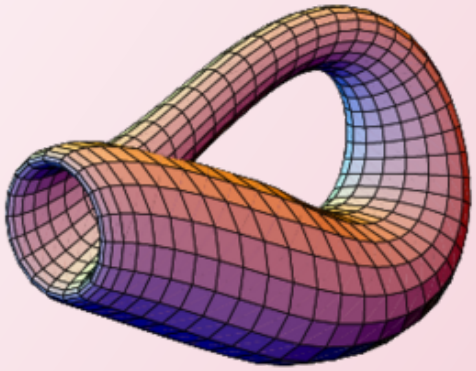


A Klein bottle (Source: Flickr)

The significance of the Klein bottle

Whilst it may just seem like a bottle, the Klein bottle's significance extends further than just a mathematical concept. It can be used to explore the boundaries of geometry, physics, and possibly even quantum physics.

Firstly, the nature of the Klein bottle forces those involved with topology to understand and explore spaces beyond



A Klein bottle in three dimensions only (Source: Creative Commons)

our three-dimensional understanding of the world and how mathematics can be used to structure objects of higher dimensions. These concepts are often used in topological paradoxes and even applied in cosmology and string theory.

Next, non-orientable surfaces like the Klein bottle have useful implications in the study of field theory and space-time mysteries, such as the bending of space time when subject to immense gravitational strength, which are also linked to theories on time travel and ways to travel through space faster than the speed of light. However, there are yet to be any discoveries related to these topics.

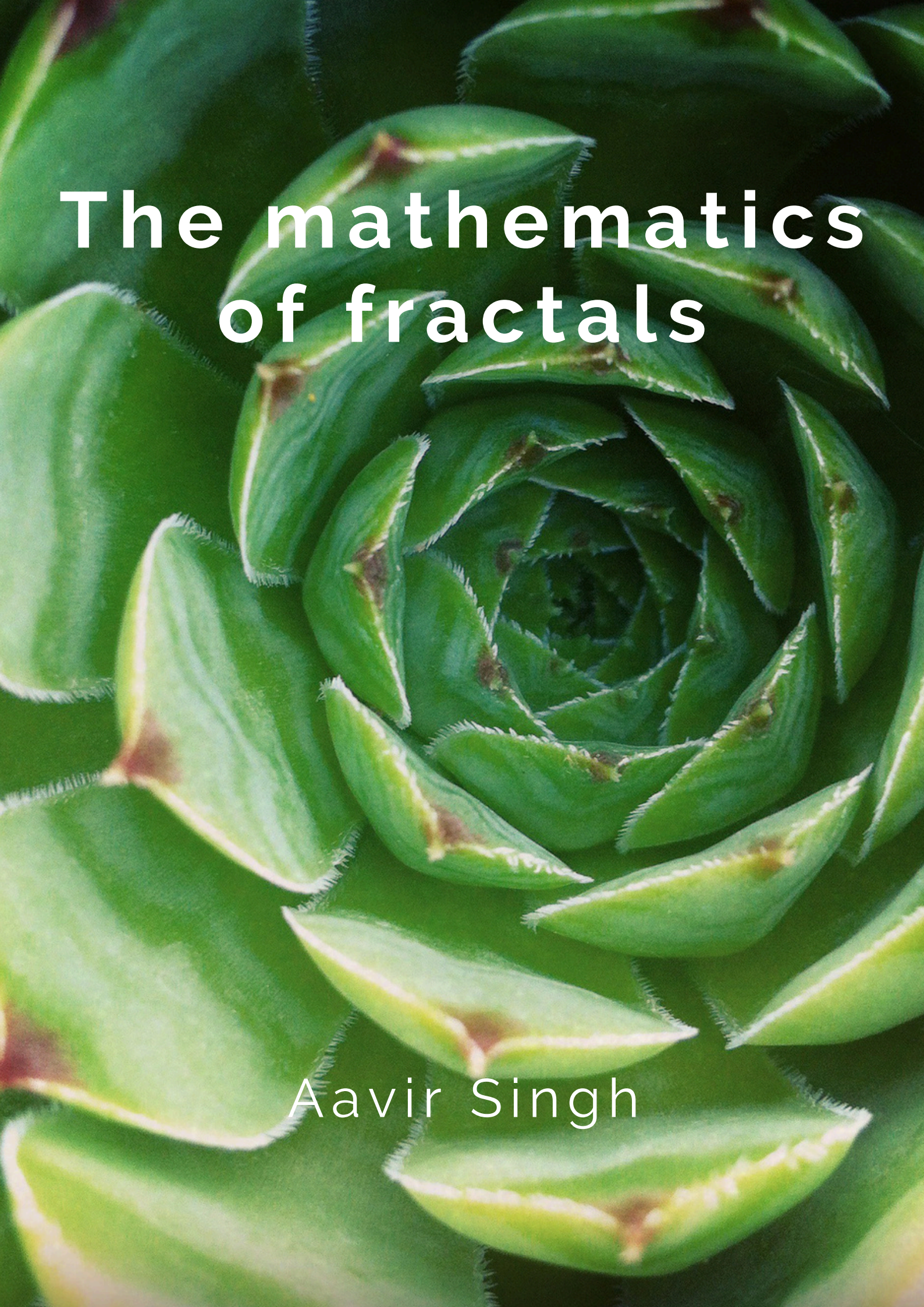
Conclusion

The Klein bottle is a single bottle that pushes the boundaries of how we understand dimensions and space. Essentially, it is a two-dimensional shape, yet it can only be modelled accurately within a four-dimensional space.

The paradoxical nature of this bottle intrigues those interested in topology around the world and raises multiple questions: is this bottle just a weird bottle, or does it hint at something much deeper about the nature of our universe and how it is composed?

"THE KLEIN BOTTLE IS A SINGLE BOTTLE THAT PUSHES THE BOUNDARIES OF HOW WE UNDERSTAND DIMENSIONS AND SPACE"

Isaac Potter



The mathematics of fractals

Aavir Singh

THE MATHEMATICS OF FRACTALS

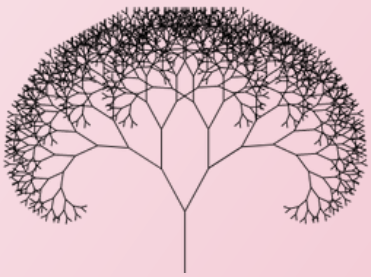
Aavir Singh

Introduction

Fractals are intricate mathematical shapes that exhibit self-similarity. In self-similarity, the magnifications of a part of a fractal resemble the overall structure or the fractal itself.

Fractals were first introduced by Benoît B. Mandelbrot in the 1970s. This led to significant transformations of the way we understand natural patterns and technological designs. From the jagged edges of coastlines to the branching of trees and lightning, fractals describe complex structures that traditional geometry struggles to model.

This article on fractals explores the maths behind fractals and their application in natural phenomena, inside of computer graphics and in other practical applications such as physics research and finance.

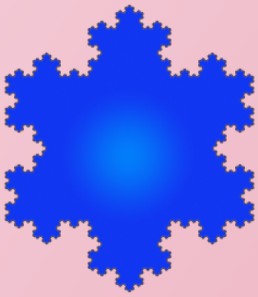


A simple fractal (Source: Creative Commons)

What are fractals?

In short, fractals are geometric shapes where each section of the shape resembles the entirety of the shape (this property is known as self-similarity). Unlike classical geometric figures, such as circles or triangles, fractals often have non-integer dimensions, called fractal dimensions, which describe their complexity. These dimensions show how fractals fill space. For instance, a line has a dimension of 1, a square has a dimension of 2, but a fractal can have a dimension of anything in between, like 1.4 or 2.8, depending on the structure it has.

One of the simplest examples of a fractal is the Koch snowflake. With each iteration, small triangles are added to every edge of a triangle, developing an infinitely complex boundary. Through this, being infinitely repeated, the perimeter of the figure would infinitely increase, while the area remains finite. This paradox illustrates how fractals defy the conventional thinking of geometry.



A Koch snowflake at sixth iteration (Source: Creative Commons)

Mathematical principles of fractals

Fractals are normally generated using iterative processes. These processes apply a rule repeatedly and each step builds on the previous. A few examples of this are:

- **Recursive algorithm**: A function is repeatedly applied to its own output, consequently producing self-similar patterns;
- **Complex Numbers**: fractals like the Mandelbrot set are generated using repetitions in the complex plane;
- **Scaling and symmetry**: Many fractals are invariant under scaling, therefore the structure remains in the same place regardless of how much you magnify or shrink the image.

Fractals in nature

Fractals are not only present in mathematical curiosities, but also are abundant in nature. Here are some examples:

- **Coastlines and mountains:** the irregularity of coastlines and mountain ranges can be modelled using fractal geometry as its complexity remains similar across different scales;
- **Plants and trees**: The branching patterns of trees, ferns and blood vessels often follow these fractal rules. This maximises efficiency in distributing resources. For example, the smaller branches of a tree could resemble the overall shape of the tree itself;
- **Clouds and rivers:** the formations of clouds and the random meanderings of rivers also reflect on fractal patterns where larger structures mimic the shapes of their smaller components.



A fractal shoreline (Source: Flickr)

Fractals in computer graphics

Fractals have emerged as necessary tools in computer graphics, functioning by rendering realistic landscapes, textures, and animations. Their ability to model complex and natural patterns makes them indispensable in industries like film, gaming and virtual reality. Key applications include:

- **Procedural generation:** fractal algorithms are used to create detailed terrains for video games and movies. This reduces the need for manual designing;
- **Image compression**: fractal-based techniques compress images by identifying repeating patterns. This is useful for images with natural scenery.
- **Special effects:** fractals help simulate realistic effects like fire, smoke and explosions by mimicking the chaotic behaviour of real-world phenomena. By applying these chaotic and unpredictable behaviours of real-life phenomena, fractals can help create visually stunning effects in films and animations.

Fractals in Physics and the Stock Market

Fractals can also be applied to physics and financial markets through the idea of complex and unpredictable systems.

In physics, fractal geometry helps model phenomena such as turbulence and the distribution of matter in the universe. Turbulent fluid flow displays patterns that can be described using fractals.

Similarly, in the world of financial markets, fractal analysis is used to model and predict stock price movements. Stock prices often follow chaotic, non-linear paths, and fractals provide a way of describing the irregular and unpredictable behaviours of these markets. By using fractal mathematics, analysts can study past market patterns and potentially forecast price changes. This doesn't mean it can be completely predicted, but it provides an insight into how fractals in finance can offer detailed views into seemingly random or complex numbers.

Limitations and challenges

Even though fractals are extremely versatile, they possess limitations. For example, modelling highly irregular structures precisely can require significant computational resources. Moreover, fractal geometry does not account for all real-world phenomena as natural systems usually always involve the factor of randomness. Furthermore, creating fractals that match specific real world data can be really difficult due to the infinite complexity of these fractals.

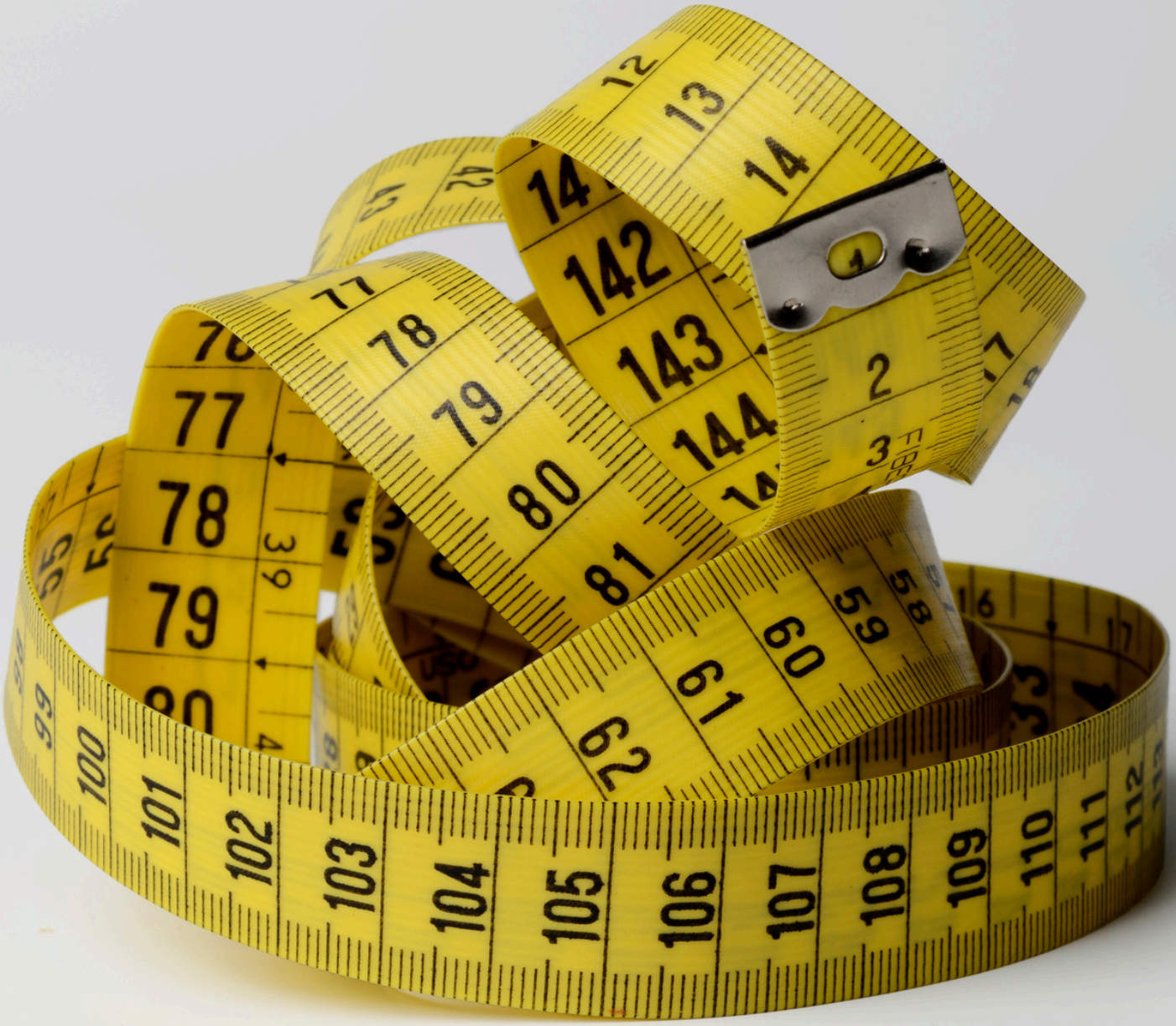
Conclusion

Fractal geometry bridges the gap between abstract mathematics and real-world applications. Fractals help us understand nature's complexity and have improved fields like physics, finance and computer graphics. Applications will increase with better understanding and increased computational tools, offering a greater understanding of the world around us.

"FRACTALS PROVIDE A WAY OF DESCRIBING THE IRREGULAR AND UNPREDICTABLE BEHAVIOURS OF [FINANCIAL] MARKETS"

Aavir Singh

Is it impossible to live in a world without mathematics?



Jacky Ng

IS IT IMPOSSIBLE TO LIVE IN A WORLD WITHOUT MATHEMATICS?

Jacky Ng

Introduction

Many people say things like, "I've never liked maths!". Some even dream of a world without it. But is this possible?

The joy that some people would have living in a world without mathematics wouldn't last long. How would we set our alarms? How would we know how much sleep we need? Would we have invented the wheel? Would the Big Bang have even happened?

Without mathematics, we couldn't understand our world, from our enormous universe, to the electron in a hydrogen atom. Between these two extremities, mathematics provides a vital part of our everyday lives in so many different ways.

Is mathematics an invention by the human brain, or a discovery of the language of nature?

This question has been around for a very long time, and even now we still do not possess a clear answer. It really depends on how a person sees it:

- A **Platonist** believes that mathematics is something we are born with;
- A **non-Platonist** believes that mathematics is something created by humans.

The importance of mathematics

Mathematics is involved in our lives from the moment our morning alarm goes off. After that, we take in a certain number of calories each day, use online discounts when we shop, spend a specific number of hours at work, and attend a set number of meetings.

Throughout the day, we take the time to drink 2-3 litres of water, dedicate an hour or so to exercise, and enjoy a legal right to a 20-minute rest break if you work for more than six hours a day. We also remember to turn off the fan and lights when we don't need them, to keep our electricity bills low. And at the end of the month, the three most important words are "salary is credited". All these activities have one thing in common: mathematics.

So, what would happen if mathematics didn't exist?



How much water do you drink each day? (Source: Flickr)

Lack of structure in science and technology

You cannot really understand science without understanding mathematics. Mathematics helps us understand nature, uncover its mysteries, and explain why things exist the way they do. For example, we need algebra to figure out chemical formulas, growth rates and genetic patterns. Science relies heavily on mathematics, so if we didn't have mathematics, we wouldn't make any new scientific discoveries. Because of mathematics, we know the Earth is round and orbits the sun in about 365 days. Mathematics also helps us estimate how big the universe is, and scientists think those numbers will keep getting bigger as they keep calculating.

No systematic measurement

Without mathematics, there would be no standardised system to measure time, distance, weight or temperature. Units such as metres, kilograms, seconds or litres are based on mathematics definitions and connections. Without these, we wouldn't have a common way to explain "how much", "how long", or "how far".

Everyday tasks like cooking or construction would be chaotic. Imagine trying to cook without measuring ingredients. Recipes would depend on unclear guesses like "a handful" or "a pinch", which would make the end product all over the place. Builders wouldn't have the correct tools to measure lengths or angles correctly, making it hard to build even a simple shelter.

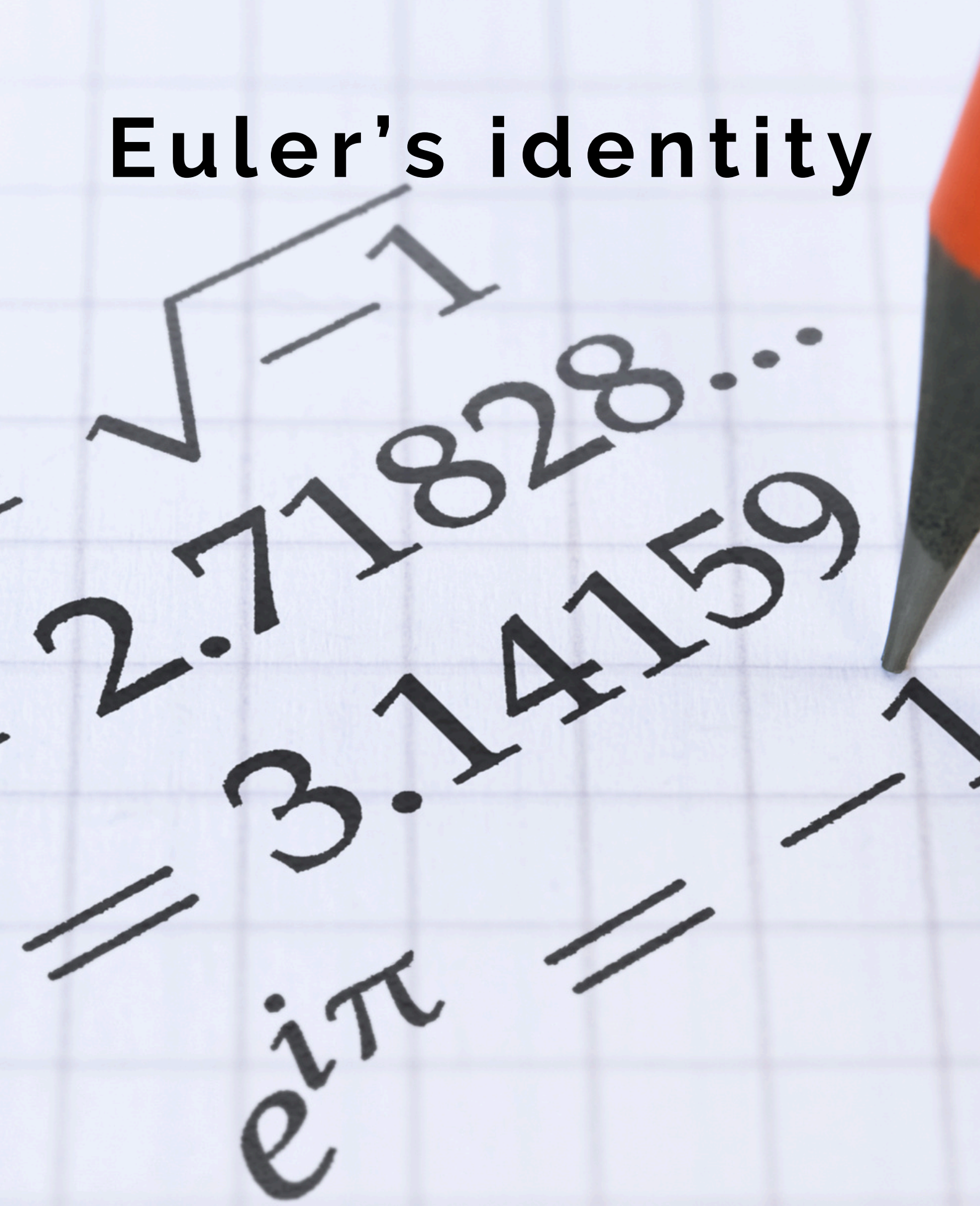
Conclusion

If there were no shapes or structures, geometry would not exist. Basic ideas like lines, circles and triangles wouldn't have clear meanings. We wouldn't be able to figure out areas or volumes. In short, the world we live in would not exist without mathematics. Tell that to the next person who insists they don't need Maths!

"SCIENCE RELIES HEAVILY ON MATHEMATICS... IF WE DIDN'T HAVE MATHEMATICS, WE WOULDN'T BE ABLE TO MAKE ANY NEW SCIENTIFIC DISCOVERIES"

Jacky Ng

Euler's identity



Saul Radford and Isabelle Eyres

EULER'S IDENTITY

Saul Radford and Isabelle Eyres

Introduction

Mathematicians often talk of the beauty of mathematics. While this may be clear to their peers, it is something often lost on the ordinary non-mathematician. However, one equation, unlike the rest, has constantly arisen as one of great beauty and wonder that can be understood by all. This is Euler's identity.

$$e^{i\pi} + 1 = 0$$

Who was Euler?

Leonhard Euler was born in 1707 and grew up in a family of mathematicians. He went on to study at the University of Basel at the age of just 13, under leading mathematicians of the time like Johann Bernoulli. During his time as a student, he published papers on various topics and went on to graduate with a Masters degree in Philosophy by the age of 16. Euler became a very devoted family man with 13 children (although only five survived to adulthood). Unfortunately, by the age of 31, Euler began to lose sight in his right eye, and by 59 he was completely blind. He died at 76, still working to the end, having published over 500 books and papers and 400 more appearing posthumously. During his working life, Euler managed to produce an average of 800 pages of work every single year.

Euler's identity

Euler's identity links five concepts at the core of mathematics:

- e (Euler's number, roughly equal to 2.718), the base of all natural logarithms;
- i, the imaginary constant defined so that i squared equals -1;
- π, the circle constant (roughly equal to 3.142);
- The numbers 1 and 0, the first positive integer and natural number respectively.

How is it that these seemingly unrelated numbers can form something so simple and astounding?

Euler's formula

Euler's identity isn't actually itself anything special. In fact, it is simply one case of another formula, that being Euler's formula, itself quite a remarkable piece of mathematics:

$$e^{ix} = cos(x) + isin(x)$$

It can then be seen that if x takes the value π, this results in Euler's identity (pi radians being equal to 180˚). But this opens up more questions than it answers. How can we know that this formula is true at all?

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The Taylor series

Euler's formula can be proved by using the Taylor series. These are series that define a certain value. To illustrate, let's find the value of e to the power of x.

First of all we will define e^x as a polynomial: $c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \dots$

If we let a = x, then all the (x-a) terms will cancel, leaving just c_0. We can therefore say that f(a) = c_0.

Let's take a similar approach with c_1. Taking the derivative of the polynomial, we get $c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$. Setting a to equal x, we get $f'(a) = c_1$.

Continuing with this pattern we get that:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)\dots$$

And by substituting in our value of e^x we get that:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\dots$$

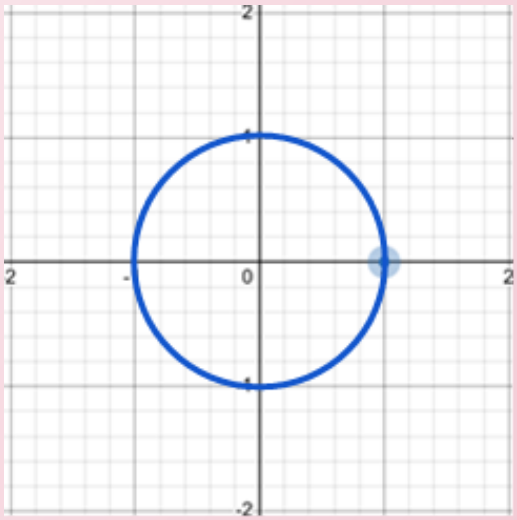
So:

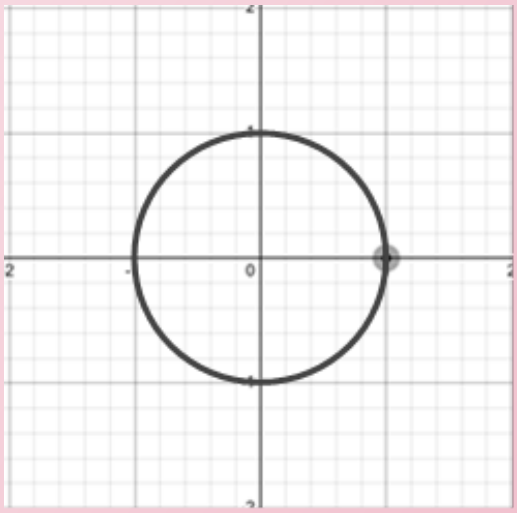
$$e^{ix} = 1 + ix + \frac{ix^2}{2!} + \frac{ix^3}{3!}\dots = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4}\dots$$

This in turn can be split into

$$e^{ix} = (1 - \frac{x^2}{2!} + \frac{x^4}{4!}\dots) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!})$$

This can be shown using a similar method to be equal to cos(x) + isin(x) as the formula states, and by simply substituting π for x we obtain the identity (pi radians being equal to 180°). Therefore, it is clear that the equation must be true, but to truly feel that the identity is true, it is better to see than to simply believe.





Not only is it clear that these two graphs of e to the power of ix (top) and cos(x) + isin(x) (bottom) are identical, but by showing them in this way, it also becomes clear why pi plays such an important role. At this value, both graphs will have travelled 180° around the origin and both be equal to -1.

The equation has uses beyond its beauty as well and was a key part of the proof of the impossibility of squaring the circle, a problem first posed by the Ancient Greeks: that creating a square with the same perimeter as a given circle was impossible.

Practical applications

Euler's constant, e, is used across a vast array of applications, from pure mathematical concepts and physics to population analysis and modern banking. Some examples of these mathematical principles include Pythagorean theorem, Riemann zeta function, the Madaurin's series, Gaussian integrals and complex analysis.

Rates of change and its applications in radioactive decay rely heavily on the use of Euler's constant. Following on from its use in physics, Euler's constant ties in very strongly with understanding waves and oscillations. It is also used widely in normal probability distribution curves in relation to IQ scores in populations. Bernoulli began to develop calculations of compound interest in 1683 using Euler's constant, and his discoveries are still applied within banking to this day.

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

Conclusion

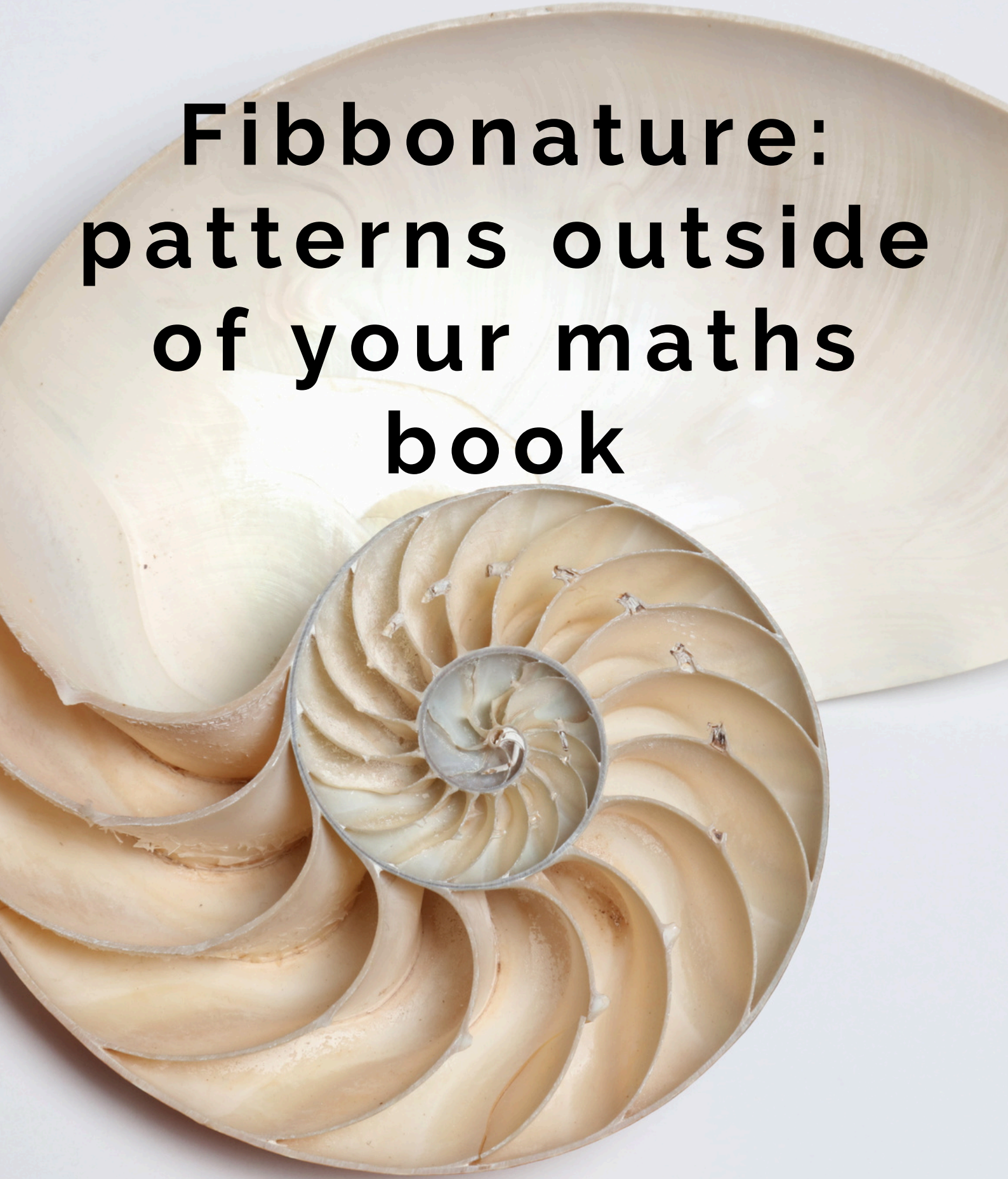
Euler's identity shows that the world of mathematics is deeply connected and that all we see around is defined by the most extraordinary mathematical laws. It is but one example of the way mathematics can continue to surprise and fascinate scholars and amateurs alike throughout the centuries.

$$e^{i\pi} + 1 = 0$$

"EULER'S CONSTANT IS USED ACROSS A VAST ARRAY OF APPLICATIONS, FROM PURE MATHEMATICAL CONCEPTS AND PHYSICS TO POPULATION ANALYSIS AND BANKING"

Isabelle Eyres

Fibbonature: patterns outside of your maths book



Maggie Dalby

FIBONNATURE: PATTERNS OUTSIDE OF YOUR MATHS BOOK

Maggie Dalby

Introduction

Fibonacci, the golden ratio, affects our everyday lives. It is almost everywhere and affects us deeply in several ways.

Fibonacci in nature

Fibonacci is the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55 etc. and it is found all over in nature. Most flowers have petals that are in the Fibonacci sequence (irises having three and roses five), and in fruits, apples have five sections and bananas three.



A sunflower with its seeds organised in a Fibonacci sequence (Source: Flickr)

However, the Fibonacci sequence is better known for being represented within the spirals in sunflower seeds. This allows the sunflower to be packed as efficiently as possible, as if it were in straight lines there would be a lot of leftover space. Spirals allow for the most space to be used, with each seed slightly angled away from the one before. (The Beauty of Maths: Fibonacci and the Golden Ratio - BBC Bitesize)

Fibonacci in art

One way to think of Fibonacci is a spiral getting wider by a factor of 1.618 every 90 degrees. These spirals are famous for being superimposed over art, such as in the works of Leonardo da Vinci and Salvador Dalí. Experts often try to explain why we find spirals so aesthetically pleasing, with one reason being that the spiral draws our eye to the centre, which is where the focus of the artwork is.

Fractals

A fractal is a never-ending pattern that is the same, no matter how zoomed in or out you go (self-similar across different scales) created by repeating a simple process in a feedback loop. These fractals are repeated everywhere, including flowers, plants, trees, clouds, mountains, rivers, waves and coastlines. In humans, they can be observed in our brain, lungs, heart, skin, blood vessels, spinal column and nervous system.

The term 'fractal' was first used by the mathematician Benoit Mandelbort in 1975. Fractal patterns are found to have relaxing effects on us, and one way we can see fractal patterns is in nature. A research study at the University of Oregon by Richard Taylor found that fractals in nature can reduce stress levels by 60 per cent, due to the fact that people are hard-wired to respond to these fractals specifically in nature.

Fractals in nature

Trees have a multitude of fractals coded into them: from their seeds to pinecones to branches and their leaves. Due to self-replication, each tree branch is an exact replica of the one before it and is seen repeatedly in the fractal structure of life forms.



Fractals grow on trees! (Source: Flickr)

Symmetry in nature

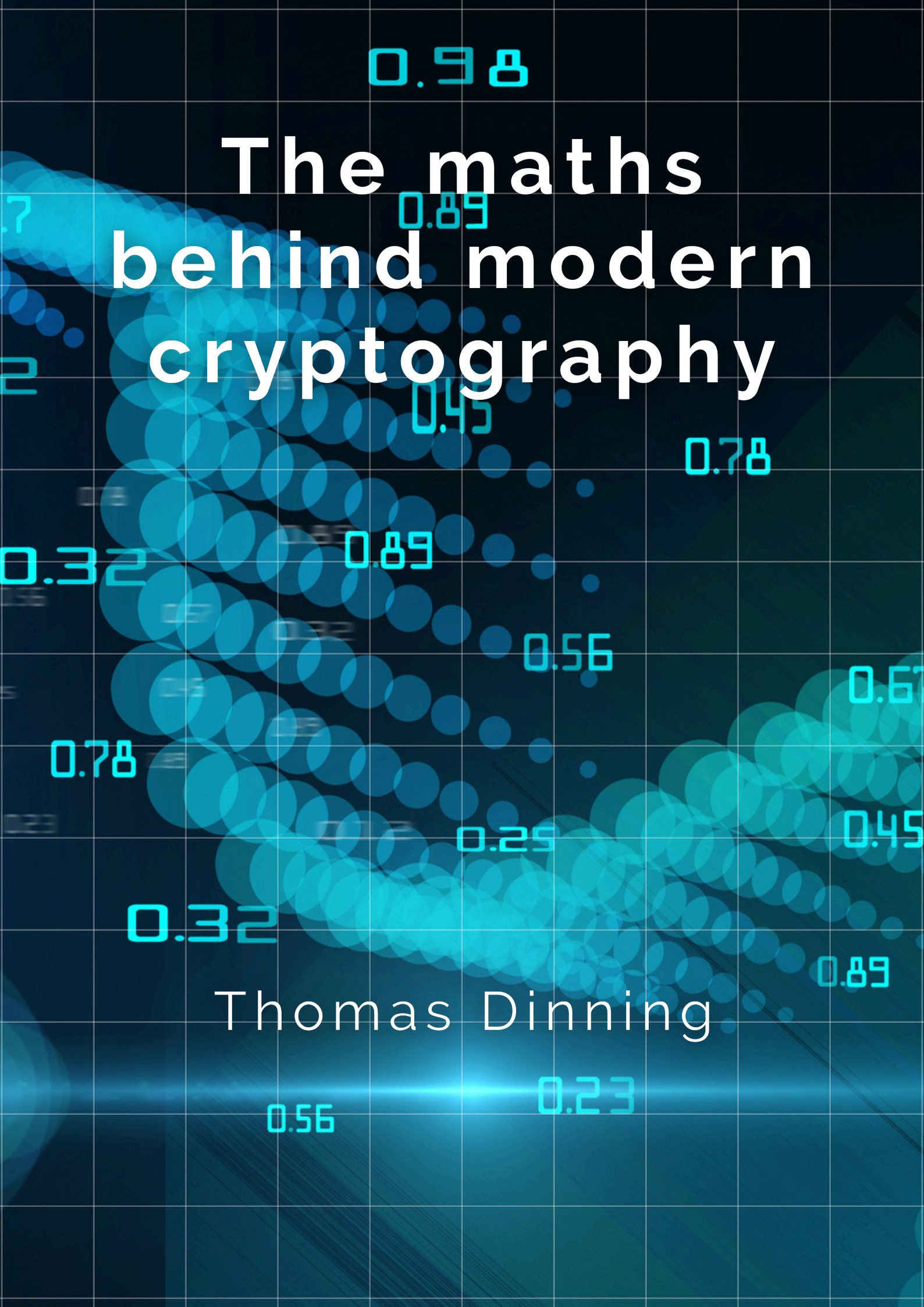
Symmetry is the property of an object where it can be split into identical parts (for example, cutting a square into two equal parts). One place that symmetry is found is in snowflakes: ice crystals formed in the atmosphere. When a dust or pollen particle meets water vapour, that then forms ice crystals on the particle creating a symmetrical patterns. Each snowflake that is created is unique from the one before. In 1591, mathematician Thomas Harriot identified the sixfold symmetry of snowflakes.

Leaves are another example of patterns in nature. While not truly symmetrical, they also always appear as Fibonacci numbers. This allows no leaf to cast shadows on another and allows them to receive the same amount of air and light, while also making sure that they cannot overlap. Therefore, all the leaves can catch the water they need to flow down to the roots for growth. Leaves have some forms of symmetry, such as bilateral and dorsiventral symmetry. This means that the leaves are identical when folded in half, but the front and back sides of the leaf are not identical.

Symmetry is also important in animal feeding. A study in a zoo found that when a tree has both symmetrical and asymmetric leaves, the ones that were eaten first were usually the symmetrical ones, showing that animals have an instinctive preference for symmetrical leaves.

Conclusion

Patterns are found everywhere in life, from fractals in your body to symmetry in leaves. The Fibonacci sequence affects the way plants are organised, and these patterns are hard-wired into our lives, from reducing our stress to animals' instinctive preference for symmetry.



The maths behind modern cryptography

Thomas Dinning

THE MATHS BEHIND MODERN CRYPTOGRAPHY

Thomas Dinning

Introduction

Originally from the Greek “Kryptos” (hidden) and “Graphia” (writing), cryptography is the study of methods of concealing data or information. The process of obfuscating the data is encryption and the reverse is known as decryption. This is unimaginably useful in all areas of life - especially in the digital age. Most data sent between computers needs to be encrypted because there is always a chance of interception by those with malicious intent. This has always been an issue, since the time of the ancient Greeks and Romans, who needed to transfer military messages securely.

The first cypher

One of the earliest known methods of encoding text is known as the Caesar cypher or Caesar shift, attributed to the Roman emperor of the same name. It works by assigning each letter in the alphabet a number from 1 to 26. When encrypting, each letter’s number has 3 added to it, so the word “Cypher” would become 3, 25, 16, 8, 5, 18 then 6, 2, 19, 11, 8, 21 (25+3 = 28, but we subtract 26 from this so the remaining number is within the 1-26 range). Converting this back to letters, we get “fbskhu”.

Cyphers in the Common Era (CE)

Development was slow, but by around 800 the Arabs had created a method to brute force substitution cyphers (where letters are swapped with each other based on a particular key) by analysing the frequency of particular characters or groups of characters and comparing them to the number of times they are expected to appear in a text, based on all the words in a particular language.

Middle Ages

In the Middle Ages, the first cyphers immune to frequency analysis were created. “Polyalphabetic cyphers” used multiple alphabets from which to substitute; this meant that the characters’ frequencies could not be found. Shortly after this, the Vigenère cypher was invented, resisting deciphering attempts until the 1800s.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
1	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s
2	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g
3	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n
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5	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
6	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r
7	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a
8	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z	a
9	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q
10	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x
11	a	b	c	d	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t	u	v	w	x	y	z
12	n	o	p	q	r	s	t	u	v	w	x	y	z	a	b	c	d	e	f	g	h	i	j	k	l	m

The Vigenère cypher (Source: Creative Commons)

Early 1900s

Huge leaps were made during the 20th century, with the first cyphering machines created and widely used in the Second World War, most notably for the Nazis’

Enigma code. The Enigma machine had dozens of parameters changed daily, and they produced wildly different codes. Rotors inside the machine would rotate after each key press, effectively encoding each position in a message with a different code.

The Digital Age

Computers hugely increased the speed of encryption as well as the complexity of codes. With a computer, data simply needs to be entered into an algorithm, which can encrypt it with an unlimited amount of steps and a huge key length.

A major flaw in all of the codes up until this point is that they are symmetric, meaning that they can be decrypted by running the encryption algorithm in reverse. This means that they are far more open to vulnerability due to the insecurity of the key. The key had to, at some point, be shared between the messenger and the receiver. This is where asymmetric encryption comes in.

Public Key Encryption

Computers store all our data, and it needs to be kept safe. Public key encryption uses different keys for encryption and decryption, meaning that anyone can be allowed access to your public key for encryption and you can decrypt the data with your private key. If you want to exchange messages with someone, you first need to exchange public keys, then you use each other's public keys to encrypt your messages to each other and your private keys to decrypt incoming data.

One of the most common methods of encryption is called RSA encryption (Rivest-Shamir-Adleman). The first step is to generate the keys. The gold standard for a final key length is 4096 bits, or 2^4096 possible numbers. This means that two prime numbers need to be generated that are 2048 bits long.



How RSA encryption works (Source: SecureW2)

Prime generation

First we generate a number, **n**, which is an odd number of the required bit length (2084 in our case). This number is then checked using the Miller-Rabin test for primes. We need to define some letters first:

- **n** is a random odd number. We can write the factors of **n-1** in the form $2^s d$. If **n** is 19, n-1 = 2^1 * 9, so **s** is 1 and **d** is 9.
- **r** is defined as a random number between 0 and **s-1**
- **a** is defined as a random number between 1 and **n-1**

If one of the following equations is true, **n** is likely prime:

$$a^d = 1 (\text{mod } n) \text{ or } a^{(2^r)*d} = -1 (\text{mod } n)$$

The first means that $a^d = k*n + 1$ so **a^d - 1** is a multiple of **n**, so if **(a^d - 1) / n** is a

whole number, **n** might be prime.

Some primes may not trigger this test, though, so we must also check whether the second equation holds up.

$(a^{(2^r)*d} + 1)/n$ should be a whole number. If the equation holds up in either case, **n** is likely to be a prime number. However, in some very select cases, these tests may return false positives for Carmichael numbers. For this reason, we always run the test multiple times with different numbers for **a** and **r** in order to mitigate this. This comes about because the test relies on properties that are always true for primes and extremely rarely true for composite numbers. This process needs to be run twice, providing us with two very large prime numbers **p** and **q**. These primes **must** be kept secret.

Public key

N = p*q. This makes **N** a composite number with two prime factors. This number is going to make up part of the public and private keys. We then find Euler's totient function, ϕ, for **n**. ϕ = (p-1)*(q-1). Now we need to find a number **e** for which 1<e<ϕ and e and ϕ have no common factors other than 1 (i.e. they are coprime, or relatively prime to each other). The public key is made up of two parts: **N** (the modulus) and **e** (the exponent). The distinct parts are used separately in encryption and can be thought of as being stored separately (although, in reality, they are stored as a single binary string).

Private key

The private key also uses **N** as its modulus. To calculate its exponent **(d)**, you just need to work out the modular inverse of **d = e mod(ϕ)** which is **d*e = 1 mod(ϕ)** and can be solved using the extended Euclidean algorithm.

Encryption

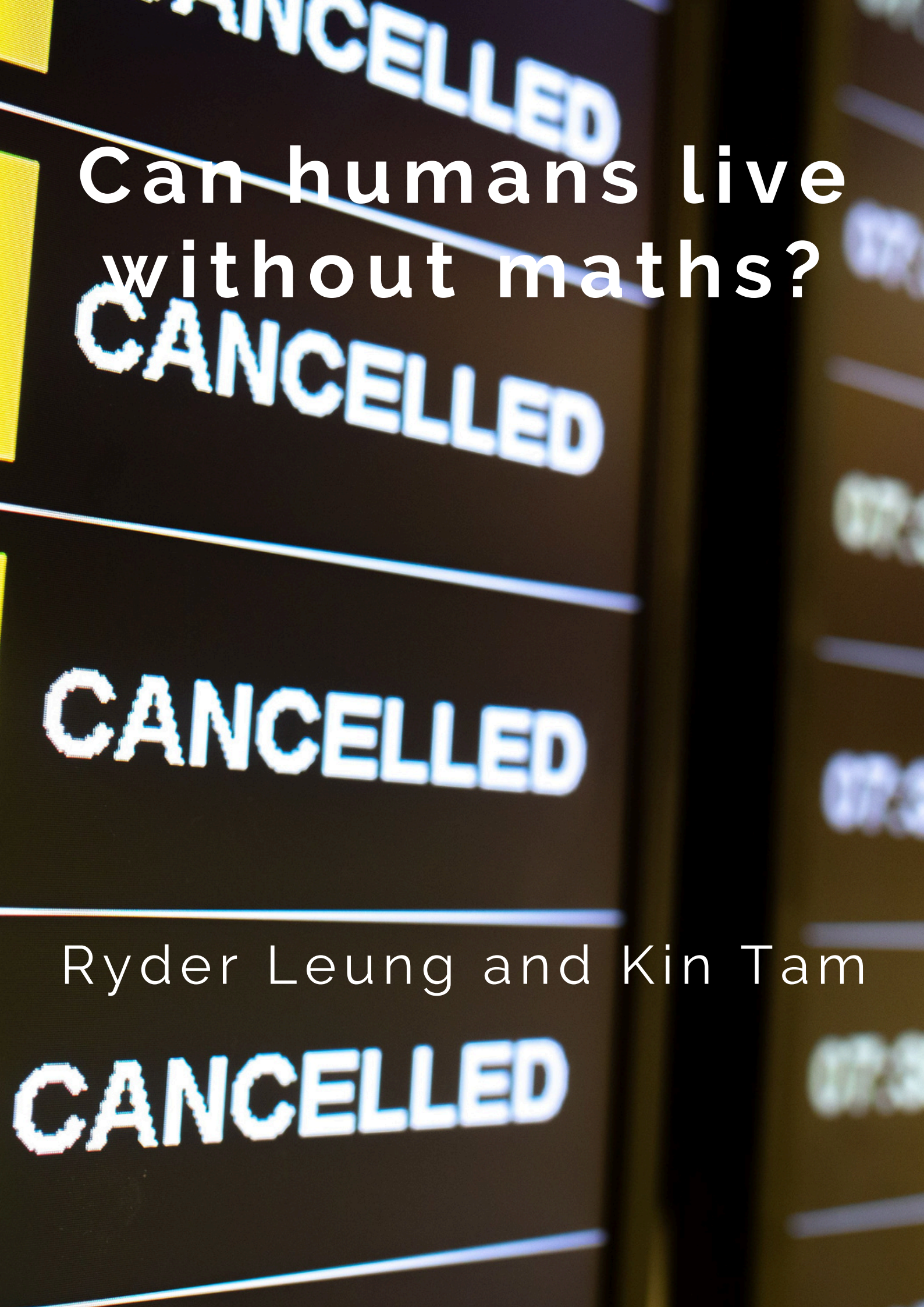
For encryption, we use the public key with parts **N** and **e**. We first convert our data into a binary number (there are many ways of doing this, including ascii and unicode for text) and add a padding as an extra layer of security. The output is **c = m^e mod(n)** where **c** is the cyphertext and **m** is the original message. This encryption can only be performed on numbers and so conversions are always needed.

Decryption

To decrypt, we run the inverse formula to that of encryption, **m = c^d mod(n).**

Conclusion

To summarise, cryptography is a study essential to everything that we do today, with so much of our data being transmitted every second. It is essential that our private information is kept secure. At its core, cryptography is ruled by mathematics. From number systems to algorithms, even going back hundreds of years, mathematicians have kept our data safe.



Can humans live without maths?

Ryder Leung and Kin Tam

CAN HUMANS LIVE WITHOUT MATHS?

Ryder Leung and Kin Tam

Introduction

You may think that maths is only used during your Maths and Science lessons and that you no longer need maths in your day-to-day life. We will explore the impact on our lives if maths is absent.

Transportation

Every public transport system has a timetable. If we don't have maths then we can't even use the most simple equation, **time = distance/speed** to calculate the time we need to go to a station. Without the correct timing, passengers will miss their public transport and be late to their destination.

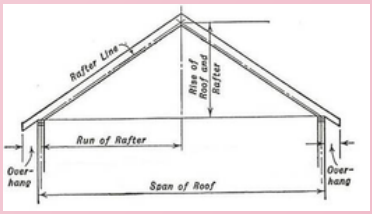
When driving, most people will aim for the most efficient path between their departure point and their destination. This could be a shortcut, with fewer sets of traffic lights or cars. Without maths, we can no longer evaluate the impact of each variable. This will greatly increase the time it takes to drive to a destination.

The consequences are even more significant when it comes to aeroplanes. Not only do aeroplanes require delicate operation, but also require precision in taking off and landing. Maintaining the correct velocity and altitude is essential for keeping the plane in the air. Both velocity and altitude are calculated using maths; without maths, the plane will not fly.

Construction

The strength of a building depends on many factors, such as the material used and the structure of the building. In order to obtain the breaking point of material, engineers use machines to compress the material until it breaks. They use maths to calculate how much force the material can endure.

The structure of a building determines the distribution of force. The main pillars and wall can handle more force than others. Maths can be used to calculate the best positions for the pillars and wall. Spreading force evenly to different supporting pillars is also important as one pillar may deteriorate rapidly if it sustains more force than others. Maths is used to calculate whether the force is evenly distributed.



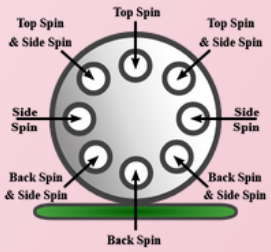
Source: jwilson

We also need to balance the strength of the material used with the price it costs. Materials like steel have high strength but it is very expensive. Some parts of buildings do not need the most expensive materials; maths can be used to calculate the most appropriate materials for each position. Without maths, all buildings would be a potential hazard and a potential waste of money.

Snooker

Snooker contains 15 red balls and six other colour balls (plus the cue ball). Each ball is worth a different number of points (the red balls are worth 1; the black ball is worth 7). In snooker, you must pot a red ball, then a ball of a different colour, then a red ball, then a ball of a different colour, and so on. Once the red balls have all been potted, you must pot the other colours in order (yellow, green, brown, blue, pink, black). The maximum "break" (potting all the balls without missing) is 147.

When playing snooker, the players need to predict where the balls will go after potting the white ball. This is where maths comes in. When the ball is potted in a different angle against the wall, the ball will follow a different path. Without the application of spin, the ball will always be reflected with the same angle as when it hit the wall. In snooker, there are three types of spin: topspin, backspin and sidespin. Each spin will have a different impact on the direction of the ball. Combined with force and friction, there are tons of calculations required to determine the type of shot required. Getting the maths wrong means the ball will go at the wrong angle, meaning a ball does not go into the pocket or worse, the player pots the cue ball or the wrong colour ball).



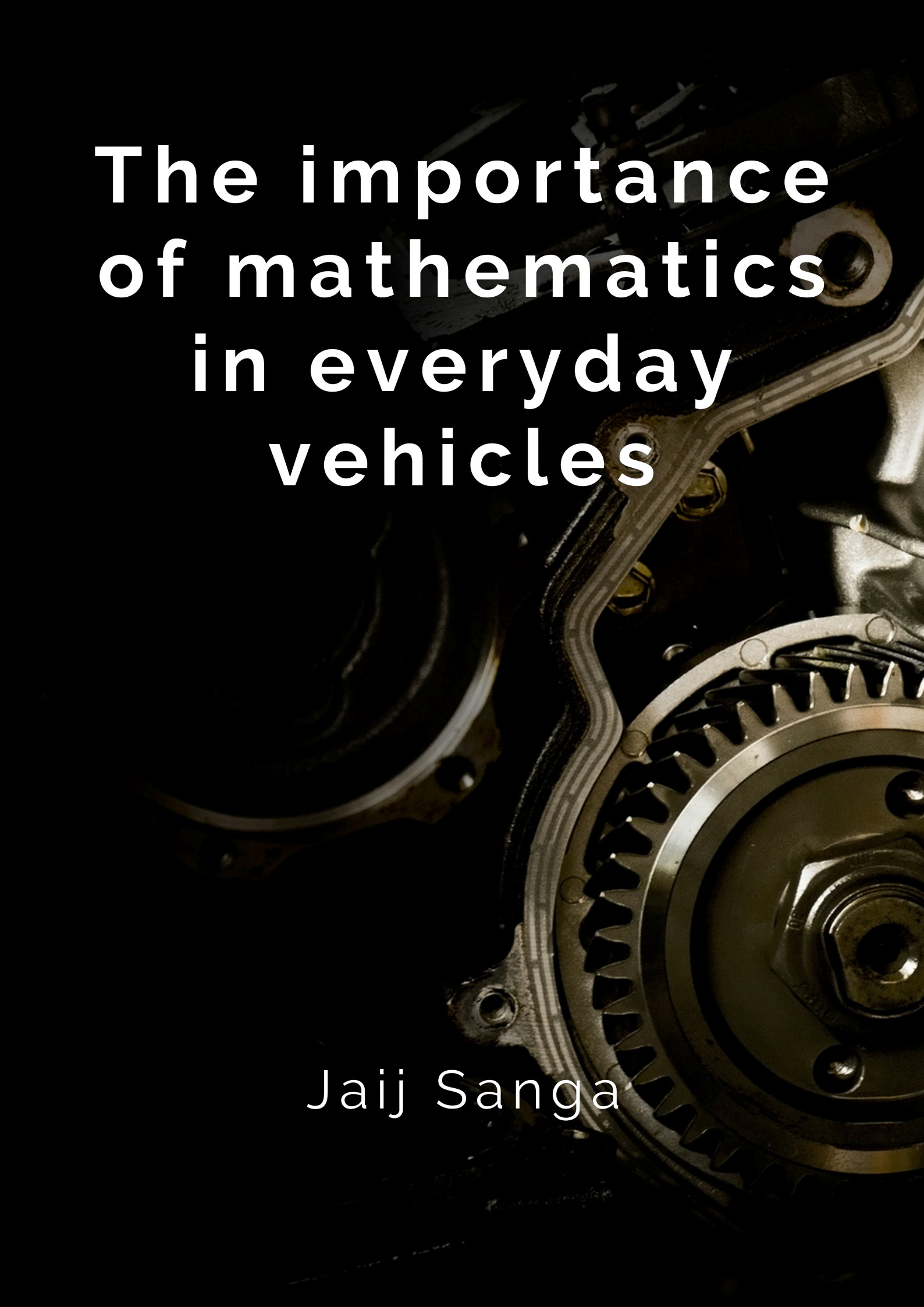
Different types of spin in snooker (Source: Creative Commons)

Conclusion

It is clear that maths plays a fundamental role in our lives, and our lives would be completely different without maths.

"COMBINED WITH FORCE AND FRICTION, [SPIN MEANS] THERE ARE TONS OF CALCULATIONS REQUIRED TO DETERMINE THE TYPE OF SHOT REQUIRED [IN SNOOKER]"

Ryder Leung and Kin Tam



The importance of mathematics in everyday vehicles

Jaij Sanga

THE IMPORTANCE OF MATHEMATICS IN EVERYDAY VEHICLES

Jaij Sanga

Introduction

Mathematics is an essential tool and application that is used in the design, operation and maintenance of the vehicles we use every day. From the most basic arithmetic, to the complex applications of force and weight ratio, mathematics plays a significant role in making sure our vehicles are efficient, safe and comfortable. Whether or not we realise it, mathematical ideas are at work every time we start the engine or put our cars in gear.

Classical motion and mechanics

The relationship between speed, distance and time influences the functioning of modern vehicles. They are all linked through the basic arithmetic equation **speed = distance/time**, which is used to find out how long it takes to cover a certain distance at constant speed. However, the exact time isn't possible to find due to external factors like speed limits, traffic and traffic lights. Nevertheless, it gives a general estimate.

Calculus is vital in mechanical engineering. It is used in many different ways, such as creating new designs and understanding the systems inside of the car. For example, when an engineer is working on a car engine, they use calculus to predict how different parts will handle heat, pressure and other factors. This helps them to build engines that are more powerful and efficient, whilst maximising safety.

The force needed to accelerate a car is given by Newton's Second Law, **F = ma**, where **F** is the force, **m** is the mass of the car, and **a** is the acceleration. This formula is central to vehicle performance, shaping everything from engine design to fuel efficiency. Furthermore, it can help to determine the speed of cars in a given time. The faster a car accelerates, the faster it will reach a higher speed - which is essential for racers.

Tyre design and traction

Mathematics is crucial in tyre design, which is important for vehicle performance and safety. Engineers use mathematical ideas in finding the best tyre size, tread patterns and material blends that ensure a vehicle will have enough traction with the road. This is to maximise safety.



The contact patch of a tyre is essential for safety (Source: rawpixel)

The contact patch of a tyre, defined as the area where the rubber comes into contact with the road, is studied using geometric modelling. The friction between the tyre and the road surface, which controls stopping distance and handling, is given by the formula **F = μN**, **F** representing the frictional force, **μ** as the coefficient of friction, and **N** as the normal force exerted by the road on the tyre. State-of-the-art tyre technology uses both calculus and differential equations to model the deformation of the tyre while in motion, in order to have optimal contact and functionality over a whole range of conditions.

Braking systems and stopping distance

Mathematics is directly involved in calculating stopping distances and in the design of braking systems. The stopping distance of a vehicle depends on a combination of factors: the initial speed of the car, the braking force, and the road surface. The basic stopping-distance equation results from physics and uses the formula: **stopping distance = thinking distance + braking distance**, or **stopping distance = $v^2/2\mu g$** (where **v** is the initial velocity of the car, and **g** represents the deceleration resulting from the application of the braking force). The related formula indicates that the stopping distance is a function of increasing velocity, where even slight increases in speed result in very large increases in stopping distances.

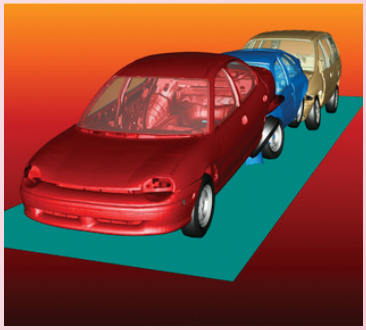
Vehicles today are equipped with more efficient braking systems, such as anti-lock braking systems (ABS), that use real-time computation methods to eliminate wheel lock-up and enhance the effectiveness of braking. ABS uses sensors and algorithms to modulate the braking force in real time, ensuring the car is slowed down effectively and safely.

"STATE-OF-THE-ART TYRE TECHNOLOGY USES BOTH CALCULUS AND DIFFERENTIAL EQUATIONS TO MODEL THE DEFORMATION OF THE TYRE WHILE IN MOTION"

Jaij Sanga

Safety features and crash simulation

In modern cars, one of the major uses of mathematics is in designing safety systems, especially in crash simulation and impact analysis. With complex mathematical models, engineers can simulate the effects of a collision on both the car and the occupants, based on physical principles. Such simulations are extremely instrumental in the design of crumple zones, airbags and the seatbelt system, all aimed at reducing the chances of injury in case of an accident. Mathematical models also forecast the forces at play in a collision and determine what materials and structural combinations will better absorb the forces in order to protect the passengers. The car structure is designed in such a manner that in the case of a head-on collision, it deforms in an organised way so that the impact energy gets absorbed. The forces exerted on the car under those circumstances can be modelled using differential equations and by the basic laws of physics. Such models allow engineers to design cars that are structurally sound but also have safety features to minimise possible injuries to the occupants.



A crash test simulation (Source: Creative Commons)

Conclusion

Maths is an essential part of our cars and our lives. It helps us be safe every time we get in or out of a car, and helps us in ways we don't even know about. We have taken considerable time and effort to ensure that cars are as efficient and safe as possible by using mathematical principles.

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